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DOCTORAL DISSERTATION

MODULI OF CONTINUITY OF QUASIREGULAR MAPPINGS

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Summary

This thesis consists of Chapters 1 and 2. The main results are contained in the two preprints and two published papers, listed below.

Chapter 1 deals with conformal invariants in the euclidean space $\mathbb{R}^n, n \geq 2$, and their interrelation. In particular, conformally invariant metrics and balls of the respective metric spaces are studied. Another theme in Chapter 1 is the study of quasiconformal maps with identity boundary values in two different cases, the unit ball and the whole space minus two points. These results are based on the two preprints:

R. Klén, V. Manojlović and M. Vuorinen: *Distortion of two point normalized quasiconformal mappings,* arXiv:0808.1219[math.CV], 13 pp.,

V. Manojlović and M. Vuorinen: *On quasiconformal maps with identity boundary values,* arXiv:0807.4418[math.CV], 16 pp.

Chapter 2 deals with harmonic quasiregular maps. Topics studied are: Preservation of modulus of continuity, in particular Lipschitz continuity, from the boundary to the interior of domain in case of harmonic quasiregular maps and quasiisometry property of harmonic quasiconformal maps. Chapter 2 is based mainly on the two published papers:

M. Arsenović, V. Kojić and M. Mateljević: *On Lipschitz continuity of harmonic quasiregular maps on the unit ball in* $\mathbf{R}^n$., Ann. Acad. Sci. Fenn. Math. 33 (2008), no. 1, 315–318.

V. Kojić and M. Pavlović: *Subharmonicity of* $|f|^p$ *for quasiregular harmonic functions, with applications*, J. Math. Anal. Appl. 342 (2008) 742-746

CHAPTER 1

Quasiconformal Mappings

1. Introduction

Conformal invariance has played a predominant role in the study of geometric function theory during the past century. Some of the landmarks are the pioneering contributions of Grötzsch and Teichmüller prior to the Second World War, and the paper of Ahlfors and Beurling [**AhB**] in 1950. These results lead to farreaching applications and have stimulated many later studies [**K**]. For instance, Gehring and Väisälä [**G3**], [**V1**] have built the theory of quasiconformal mappings in $\mathbb{R}^n$ based on the notion of the modulus of a curve family introduced in [**AhB**].

In the first chapter of this dissertation our goal is to study two kinds of conformally invariant extremal problems, which in special cases reduce to problems due to Grötzsch and Teichmüller, resp. These two classical extremal problems are extremal problems for moduli of ring domains. The Grötzsch and Teichmüller rings are the extremal rings for extremal problems of the following type, which were first posed for the case of the plane. Among all ring domains which separate two given closed sets E_1 and E_2, $E_1 \cap E_2 = \emptyset$, find one whose module has the greatest value.

In the general case these extremal problems lead to conformal invariants $\lambda_G(x, y)$ and $\mu_G(x, y)$ defined for a domain $G \subset \mathbb{R}^n$ and $x, y \in G$. A basic fact is that $\lambda_G(x, y)^{1/(1-n)}$ and $\mu_G(x, y)$ are metrics. Following closely the ideas developed in [**Vu1**] and [**Vu2**] we study three topics: (a) the geometry of the metric spaces (G, d) when d is $\lambda_G(x, y)^{1/(1-n)}$ or $\mu_G(x, y)$, (b) the relations of these two metrics to several other metrics and (c) the behavior of quasiconformal mappings with respect to several of these metrics. One of our main results is to present a revised version of the Chart on p. 86 of [**Vu1**], taking into account some later developments, such as [**H**], [**HV**], [**Vu2**].

Then we present an application to the geometry of balls in these metrics. As a special case we investigate λ metric in $B^2 \setminus \{0\}$, continuing work of [**H**].

Another question we address is: if $f : (G_i, m_{G_i}) \longrightarrow (G'_i, m_{G'_i})$ is uniformly continuous ($i = 1, 2$), is $(G, m_G) \longrightarrow (G', m_{G'})$ uniformly continuous ($G = G_1 \cup G_2$, $G' = G'_1 \cup G'_2$)?

Chapter 1 concludes with displacement estimates for K-qc mappings which are identity on the boundary of G.

In the second chapter we explore what additional information on a K-qc mappings we get if we assume it is also harmonic. We call such mappings hqc-mappings.

In case $n = 2$ we show that hqc map has the same type of moduli of continuity on $\overline{D}$ as on ∂D.

A similar, for the Lipschitz case, result is proved on B^n. Finally, we show, for $n = 2$, that any hqc map is bilipschitz in quasihyperbolic metric.

2. The extremal problems of Grötzsch and Teichmüller

In what follows, we adopt the standard definitions notions related of quasiconformal mappings from **[V1]**.

We use notation $B^n(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, $S^{n-1}(x,r) = \{y \in \mathbb{R}^n : |x-y| = r\}$, $H^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\}$ and abbreviations $B^n(r) = B^n(0,r)$, $B^n = B^n(1)$, $S^{n-1}(r) = S^{n-1}(0,r)$ and $S^{n-1} = S^{n-1}(1)$.

For the modulus $M(\Gamma)$ of a curve family Γ and its basic properties we refer the reader to **[V1]**. Its basic property is conformal invariance.

For $E, F, G \subset \overline{\mathbb{R}}^n$ let $\Delta(E,F,G)$ be the family of all closed curves joining E to F within G. More precisely, a path $\gamma : [a,b] \to \overline{\mathbb{R}}^n$ belongs to $\Delta(E,F,G)$ iff $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

If G is a proper subdomain of $\overline{\mathbb{R}^n}$, then for $x, y \in G$ with $x \neq y$ we define

$$\lambda_G(x,y) = \inf_{C_x, C_y} M(\Delta(C_x, C_y; G)) \tag{2.1}$$

where $C_z = \gamma_z[0,1)$ and $\gamma_z : [0,1) \longrightarrow G$ is a curve such that $\gamma_z(0) = z$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$. This conformal invariant was introduced by J. Ferrand (see **[Vu2]**).

For $x \in \mathbb{R}^n \setminus \{0, e_1\}$, $n \geqslant 2$, define

$$p(x) = \inf_{E,F} M(\Delta(E,F)), \tag{2.2}$$

where the infimum is taken over all pairs of continua E and F in $\overline{\mathbb{R}^n}$ with $0, e_1 \in E$, $x, \infty \in F$. This extremal quantity was introduced by O. Teichmüller (see **[Vu2]**, **[HV]**).

For a proper subdomain G of $\overline{\mathbb{R}^n}$ and for all $x, y \in G$ define

$$\mu_G(x,y) = \inf_{C_{xy}} M(\Delta(C_{xy}, \partial G; G)) \tag{2.3}$$

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0,1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$. For the case $G = B^n$ the function $\mu_{B^n}(x,y)$ is the extremal quantity of H. Grötzsch (see **[Vu2]**).

Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$ be a continuous mapping. Then we say that f is uniformly continuous if there exists an increasing continuous function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ and $d_2(f(x), f(y)) \leq \omega(d_1(x,y))$ for all $x, y \in X$. We call the function ω the modulus of continuity of f. If there exist $C, \alpha > 0$ such that $\omega(t) \leq Ct^\alpha$ for all $t > 0$, we say that f is Hölder-continuous with Hölder exponent α. If $\alpha = 1$, we say that f is Lipschitz with the Lipschitz constant C or simply C-Lipschitz. If f is a homeomorphism and both f and f^{-1}

are C-Lipschitz, then f is C-bilipschitz or C-quasiisometry and if $C = 1$ we say that f is an isometry. These conditions are said to hold locally, if they hold for each compact subset of X .

A very special case of these are isometries.

Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $f : X_1 \to X_2$ be a homeomorphism. We call f an *isometry* if $d_2(f(x), f(y)) = d_1(x, y)$ for all $x, y \in X_1$.

In this section we introduce five types of metrics:

(1) Spherical (chordal) metric q.
(2) Quasihyperbolic metric k_G of a domain $G \subset \mathbb{R}^n$.
(3) A metric j_G closely related to k_G.
(4) Seittenranta's metric δ_G.
(5) Apollonian metric α_G.

The first one is defined on $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. The second and the third ones are defined in any proper subdomain $G \subset \mathbb{R}^n$, both of them generalize hyperbolic metric (on B^n or H^n) to arbitrary proper subdomain $G \subset \mathbb{R}^n$. Seittenranta's metric is natural, Möbius invariant analogue of the j_G-metric. Apollonian metric is defined in any proper subdomain $G \subseteq \mathbb{R}^n$ which boundary is not a subset of a circle or a line.

2.4. The spherical metric. The metric q is defined by

$$q(x,y) = \begin{cases} \dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x \neq \infty \neq y, \\ \dfrac{1}{\sqrt{1+|x|^2}}, & y = \infty. \end{cases} \tag{2.5}$$

Absolute (cross) ratio of an ordered quadruple a, b, c, d of distinct points in $\overline{\mathbb{R}^n}$ is defined

$$|a,b,c,d| = \frac{q(a,c)\, q(b,d)}{q(a,b)\, q(c,d)}. \tag{2.6}$$

Now we introduce distance ratio metric or j_G*-metric.* For an open set $G \subset \mathbb{R}^n$, $G \neq \mathbb{R}^n$ we define $d(z) = d(z, \partial G)$ for $z \in G$ and

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right) \tag{2.7}$$

for $x, y \in G$.

For a nonempty $A \subset G$ we define the j_G-diameter of A by

$$j_G(A) = \sup\{j_G(x,y) \,|\, x, y \in A\}.$$

For an open set $G \subset \mathbb{R}^n$, $G \neq \mathbb{R}^n$, and a nonempty $A \subset G$ such that $d(A, \partial G) > 0$ we define

$$r_G(A) = \frac{d(A)}{d(A, \partial G)}.$$

If $\rho(x) > 0$ for $x \in G$, ρ is continuous and if γ is a rectifiable curve in G, then we define

$$l_\rho(\gamma) = \int_\gamma \rho\, ds.$$

The Euclidean length of a curve γ is denoted by $l(\gamma)$.

Also, for $x_1, x_2 \in G$ we define

$$d_\rho(x, y) = \inf l_\rho(\gamma), \tag{2.8}$$

where the infimum is taken over all rectifiable curves from x_1 to x_2.

It is easy to show that d_ρ is a metric in G.

Now we take any proper domain $G \subset \mathbb{R}^n$ and set $\rho(x) = \frac{1}{d(x,\partial G)}$.

The corresponding metric, denoted by k_G, is called the *quasihyperbolic metric* in G. Since,

$$\rho(\varphi(x)) = \frac{1}{d(\varphi(x), \partial\,(\varphi G))} = \frac{1}{d(x, \partial G)} = \rho(x),$$

for Euclidean isometry φ,

$$k_{G'}(x', y') = k_G(x, y), \qquad \text{where } G' = \varphi(G),\ \ x' = \varphi(x),\ \ y' = \varphi(y).$$

Now we introduce *Seittenranta's metric* δ_G [**Se**]. For more details on Möbius transformations in $\mathbb{R}^n$ see [**B1**]. For an open set $G \subset \mathbb{R}^n$ with card$\partial G \geqslant 2$ we set

$$m_G(x, y) = \sup_{a,b\in\partial G} |a, x, b, y|$$

and

$$\delta_G(x, y) = \log(1 + m_G(x, y))$$

for all $x, y \in G$.

Consider now the case of an unbounded domain $G \subset \mathbb{R}^n, \infty \in \partial G$. Note that if a or b in the supremum equals infinity, then we get exactly j_G metric. This implies that we always have $j_G \leqslant \delta_G$.

We will also use *Apollonian metric* studied by Beardon [**B2**], (also see [**AVV**, 7.28 (2)]) defined in open proper subsets $G \subset \mathbb{R}^n$ by

$$\alpha_G(x, y) = \sup_{a,b\in\partial G} \log |a, x, y, b| \quad \text{for all } x, y \in G.$$

This formula defines a metric iff $\mathbb{R}^n \setminus G$ is not contained in an $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

In general, the hyperbolic-type metrics can be divided into length-metrics, defined by means of integrating a weight function and point-distance metric.

Another group may again be classified by the number of boundary points used in there's definition. So for instance, the j metric is one-point metric, while the Apollonian metric is two-point metric.

2.9. Definition. A domain $A \subset \overline{\mathbb{R}^n}$ is a ring if $C(A)$ has exactly two components, where $C(A)$ denotes the complement of $A \subset \mathbb{R}^n$.

If the components of $C(A)$ are C_0 and C_1, we denote $A = R(C_0, C_1)$, $B_0 = C_0 \cap \overline{A}$ and $B_1 = C_1 \cap \overline{A}$. To each ring $A = R(C_0, C_1)$, we associate the curve family $\Gamma_A = \Delta(B_0, B_1, A)$ and the modulus of A is defined by $\mod(A) = M(\Gamma_A)$. Next, the capacity of A is by definition $\mathrm{cap} A = \omega_{n-1}(\mod A)^{1-n}$.

The complementary components of the Grötzsch ring $R_{G,n}(s)$ in $\mathbb{R}^n$ are $\overline{B}^n$ and $[s \cdot e_1, \infty]$, $s > 1$, while those of the Teichmüller ring $R_{T,n}(t)$ are $[-e_1, 0]$ and $[t\, e_1, \infty]$, $t > 0$. We shall need two special functions $\gamma_n(s)$, $s > 1$, and $\tau_n(t)$, $t > 0$, to designate the moduli of the families of all those curves which connect the complementary components of the Grötzsch and Teichmüller rings in $\mathbb{R}^n$, respectively.

$$\gamma_n(s) = M(\Gamma_s) = \gamma(s), \quad \Gamma_s = \Gamma_{R_{G,n}}(s),$$
$$\tau_n(t) = M(\Delta_t) = \tau(t), \quad \Delta_t = \Gamma_{R_{T,n}}(t).$$

These functions are related by a functional identity [**G1**, Lemma 6]

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1). \tag{2.10}$$

2.11. Definition. Given $r > 0$, we let $R\Psi_n(r)$ be the set of all rings $A = R(C_0, C_1)$ in $\overline{\mathbb{R}^n}$ with the following properties:

(1) C_0 contains the origin and a point a such that $|a| = 1$.
(2) C_1 contains ∞ and a point b such that $|b| = r$.

Teichmüller first considered the following quantity in the planar case ($n = 2$):

$$\tau_n(r) = \inf M(\Gamma_A) = \inf\{p(x) \,|\, |x| = r\},$$

where the infimum is taken over all rings $A \in R\Psi_n(r)$ and $p(x)$ is as in (2.2). For $n \geqslant 3$ it was studied in [**G1**] and in [**HV**].

2.12. Theorem. [**V1**, Theorem 11.7] *The function $\tau_n : (0, \infty) \to (0, \infty)$ has the following properties:*

(1) *τ_n is decreasing,*
(2) $\lim_{r\to\infty} \tau_n(r) = 0$,
(3) $\lim_{r\to 0} \tau_n(r) = \infty$,
(4) *$\tau_n(r) > 0$ for every $r > 0$.*

Moreover, $\tau_n : (0, \infty) \to (0, \infty)$ and $\gamma_n : (1, \infty) \to (0, \infty)$ are homeomorphisms.

From the definition of τ_n and from the conformal invariance of the modulus, we obtain the following estimate:

2.13. Theorem. *Suppose that $A = R(C_0, C_1)$ is a ring and that $a, b \in C_0$ and $c, \infty \in C_1$. Then*

$$M(\Gamma_A) \geqslant \tau_n\left(\frac{|c-a|}{|b-a|}\right).$$

Here equality holds for the Teichmüller ring, when $a = 0, b = -e_1, c = te_1, t > 0$ and $C_0 = [-e_1, 0], C_1 = [te_1, \infty)$.

2.14. Theorem. *Let $C \subset B^n$ be a connected compact set containing* 0 *and* x*, where* $|x| < 1$*. Then the capacity of a ring domain with components* $C_0 = C$*,* $C_1 = \{x : |x| \geqslant 1\}$ *is at least* $\gamma_n(\frac{1}{|x|})$*. Here equality holds for the ring with the complementary components* $[0, |x|e_1]$ *and* $\mathbb{R}^n \setminus B^n$*.*

These theorems state the extremal properties of the Teichmüller and Grötzsch rings and their proofs are based on the symmetrization theorem in [**G1**, Theorem 1].

3. Moduli of continuity

In this section we investigate the moduli of continuity of the identity mappings $id_G : (G, \rho) \longrightarrow (G, d)$ where ρ and d are chosen from the set of interesting metrics defined on G (like quasihyperbolic metric k, modulus metric μ etc.).

Hence, we are interested in results of type

$$d(x, y) \leqslant \zeta(\rho(x, y)) = \zeta_\rho^d(\rho(x, y)), \quad x, y \in G. \tag{3.1}$$

We give several estimates of this type, and then we collect these results in a charts at the end of this section.

Note that in our charts we have λ_G^{-1}, as well as in the inequalities of type (3.1); however reader should be aware that in general λ_G^{-1} is not a metric. In fact $\lambda_G^{1/1-n}$ is always a metric. For more details on this matter see [**Vu4**].

It is well known that $j_G(x, y) \leqslant k_G(x, y)$, so $\zeta_k^j(t) = t$.

3.2. Lemma. *For* $x, y \in G$

$$k_G(x, y) \geqslant \log\left(1 + \frac{m(x, y)}{\min\{d(x), d(y)\}}\right) \geqslant j_G(x, y).$$

where $m(x, y) = \inf\{l(\gamma) \,|\, \gamma$ *is a curve joining* x *and* y *in* $G\}$*.*

Proof. We may assume $0 < d(x) \leqslant d(y)$. Choose a rectifiable arc $\gamma : [0, s] \to G$ from x to y, parametrized by arc length:

$$\gamma(0) = x, \qquad \gamma(s) = y;$$

obviously $s \geqslant |x - y|$. For any $0 \leqslant t \leqslant s$ we have

$$d(\gamma(t)) \leqslant d(x) + t, \qquad \text{(a key observation)},$$

so,

$$l_\rho(\gamma) \geqslant \int_0^s \frac{dt}{d(x) + t} = \log\frac{d(x) + s}{d(x)} \geqslant \log\frac{d(x) + |x - y|}{d(x)} = j_G(x, y).$$

□

The reverse inequality is not true in general; domain G such that there is a constant $c > 0$ such that $k_G \leqslant c\, j_G$ is called uniform domain, so in that case $\zeta_j^k(t) = ct$.

3.3. Lemma. [**Vu1**, Lemma 2.21] *Let G be a proper subdomain of $\mathbb{R}^n$. If $x \in G$, $d(x) = d(x, \partial G)$ and $y \in B^n(x, d(x)) = B_x$, $x \neq y$, then*

$$\lambda_G(x,y) \geqslant \lambda_{B_x}(x,y) \geqslant c_n \log\left(\frac{d(x)}{|x-y|}\right) \tag{3.4}$$

where c_n is the positive number in [**V1**, (10.11)]. *There exists a strictly increasing function $h_1 : (0,+\infty) \longrightarrow (0,+\infty)$ with $\lim_{t\to 0_+} h_1(t) = 0$ and $\lim_{t\to+\infty} h_1(t) = +\infty$, depending only on n, such that*

$$\lambda_G(x,y) \leqslant h_1\left(\frac{\min\{d(x), d(y)\}}{|x-y|}\right) \tag{3.5}$$

for $x, y \in G$, $x \neq y$. If $x \in G$ and $y \in B^n(x, d(x)) = B_x$, $x \neq y$, then

$$\mu_G(x,y) \leqslant \mu_{B_x}(x,y) = capR_G\left(\frac{d(x)}{|x-y|}\right) \leqslant \omega_{n-1}\left(\log\left(\frac{d(x)}{|x-y|}\right)\right)^{1-n}. \tag{3.6}$$

From (3.6) we get $\mu_G(x,y) \leqslant \gamma\left(\frac{d(x)}{|x-y|}\right)$ for $x \in G$ and $y \in B_x$. It is equivalent with $\mu_G(x,y) \leqslant \gamma\left(\frac{1}{r}\right)$ where $r = \frac{|x-y|}{d(x)}$.

We can express $j_G(x,y)$ in terms of r: $r = e^j - 1$ and obtain

$$\mu_G(x,y) \leqslant \gamma\left(\frac{1}{e^j - 1}\right).$$

This gives $\zeta_j^\mu(t) = \gamma\left(\frac{1}{e^t-1}\right)$ locally.

3.7. Lemma. [**Vu1**, Lemma 2.39] *For $n \geqslant 2$ there exists strictly increasing function $h_2 : [0,+\infty) \longrightarrow [0,+\infty)$ with $h_2(0) = 0$ and $\lim_{t\to+\infty} h_2(t) = +\infty$ with the following properties.*

If E is closed and F is compact in $\mathbb{R}^n$ then

$$M(\Delta(E,F)) \leqslant h_2(T); \quad T = \min\{j_{\mathbb{R}^n\setminus E}(F), j_{\mathbb{R}^n\setminus F}(E)\}. \tag{3.8}$$

In particular, if G is a proper subdomain of $\mathbb{R}^n$, then

$$\mu_G(x,y) \leqslant h_2(3k_G(x,y)) \tag{3.9}$$

for all $x, y \in G$. Moreover, there are positive numbers b_1, b_2 depending only on n such that

$$\mu_G(x,y) \leqslant b_1 k_G(x,y) + b_2 \tag{3.10}$$

for all $x, y \in G$.

From (3.9) we have $\zeta_k^\mu(t) = h_2(3t)$.

3.11. Lemma. [**Vu1**, Lemma 2.44] *If $E, F \subseteq \mathbb{R}^n$ are disjoint continua, then*

$$M(\Delta(E,F)) \geqslant \bar{c}_n \min\{j_{\mathbb{R}^n\setminus E}(F), j_{\mathbb{R}^n\setminus F}(E)\}$$

where $\bar{c}_n$ is a positive number depending only on n.

3.12. Corollary. [**Vu1**, Corollary 2.46] *If E and F are disjoint continua in $\overline{\mathbb{R}^n}$ and $\infty \in F$, then*

$$M(\Delta(E,F)) \geqslant c_n j_{\mathbb{R}^n \setminus F}(E).$$

3.13. Corollary. [**Vu4**, Lemma 6.23] *Let $G \subseteq \mathbb{R}^n$ be a domain $G \neq \mathbb{R}^n$ and connected boundary ∂G. Then*

$$\mu_G(a,b) \geqslant c_n j_G(a,b) \tag{3.14}$$

holds for $a, b \in G$. If, in addition, G is uniform, then

$$\mu_G(a,b) \geqslant B\, k_G(a,b) \tag{3.15}$$

for all $a, b \in G$.

The first part of this corollary gives $\zeta_\mu^j(t) = \frac{1}{c_n} t$ if ∂G is connected. (3.15) gives $\zeta_\mu^k(t) = c\,t$ if ∂G is connected and G is uniform.

3.16. Lemma. [**AVV**, Corollary 15.13]*Let G be a proper subdomain of $\mathbb{R}^n$, x and y distinct points in G and $m(x,y) = \min\{d(x), d(y)\}$. Then*

$$\lambda_G(x,y) \leqslant \sqrt{2}\tau\left(\frac{|x-y|}{m(x,y)}\right). \tag{3.17}$$

From (3.17) using again $r = e^j - 1$, $r = \frac{|x-y|}{m(x,y)}$, we have

$$\sqrt{2}\tau(e^j - 1) \geqslant \lambda_G,$$

and then, since τ is decreasing, $e^j \leq \tau^{-1}\left(\frac{\lambda_G}{\sqrt{2}}\right)$ and from here

$$j \leqslant \log\left(1 + \tau^{-1}\left(\frac{1}{\sqrt{2}\,\lambda_G^{-1}}\right)\right).$$

Finally we obtain $\zeta_{\lambda^{-1}}^j(t) = \log\left(1 + \tau^{-1}\left(\frac{1}{\sqrt{2}\,t}\right)\right)$.

3.18. Definition. *A closed set E in $\mathbb{R}^n$ is called a c-quasiextremal distance set or c-QED exceptional or c-QED set, $c \in (0,1]$, if for each pair of disjoint continua $F_1, F_2 \subseteq \overline{\mathbb{R}^n} \setminus E$*

$$M(\Delta(F_1, F_2; \overline{\mathbb{R}^n} \setminus E)) \geqslant c M(\Delta(F_1, F_2)). \tag{3.19}$$

If G is a domain in $\overline{\mathbb{R}^n}$ such that $\overline{\mathbb{R}^n} \setminus G$ is a c-QED set, then we call G a c-QED domain.

3.20. Theorem. [**Vu3**, Theorem 6.21] *Let G be a c-QED domain in $\mathbb{R}^n$. Then*

$$\lambda_G(x,y) \geqslant c\tau(s^2 + 2s) \geqslant 2^{1-n} c\tau(s) \tag{3.21}$$

where $s = \frac{|x-y|}{min(d(x),d(y))}$.

From the first inequality in (3.21), taking into account $s = e^j - 1$, we obtain

$$\lambda^{-1} = \frac{1}{\lambda} \leqslant \frac{1}{c}\,\frac{1}{\tau((s+1)^2-1)} = \frac{1}{c}\,\frac{1}{\tau(e^{2j}-1)}.$$

This gives $\zeta_j^{\lambda^{-1}}(t) = \dfrac{1}{c}\,\dfrac{1}{\tau(e^{2t}-1)}$ for c-QED domain G.

Combining ζ_k^j and $\zeta_j^{\lambda^{-1}}$ we estimate λ_G^{-1} in terms of k_G, so $\zeta_k^{\lambda^{-1}} = \zeta_j^{\lambda^{-1}} \circ \zeta_k^j = \zeta_j^{\lambda^{-1}}$. In fact, we have

$$\lambda_G^{-1} \leqslant \frac{1}{c}\frac{1}{\tau(e^{2j}-1)} \leqslant \frac{1}{c\tau(e^{2k}-1)}.$$

The fields $\zeta_\mu^{\lambda^{-1}}$, $\zeta_{\lambda^{-1}}^k$, $\zeta_{\lambda^{-1}}^\mu$ are obtained in the same fashion as $\zeta_k^{\lambda^{-1}}$, namely as compositions of appropriate functions ζ_ρ^d. We use following inequalities.

For $\zeta_\mu^{\lambda^{-1}}$ we have

$$\lambda_G^{-1} \leqslant \frac{1}{c\tau(e^{2j}-1)} \leqslant \frac{1}{c\tau(e^{2\mu/c_n}-1)} = \frac{1}{c\tau(e^{b\mu}-1)},$$

where the second inequality follows from (3.14) and where $b = \frac{2}{c_n}$.

For $\zeta_{\lambda^{-1}}^k$ we have

$$k_G \leqslant c\, j_G \leqslant c \log\left(1 + \tau^{-1}\left(\frac{1}{\sqrt{2}\,\lambda_G^{-1}}\right)\right)$$

and for $\zeta_{\lambda^{-1}}^\mu$ we have

$$\mu_G \leqslant \gamma\left(\frac{1}{e^j - 1}\right) \leqslant \gamma\left(\frac{1}{e^{\log\left(1+\tau^{-1}\left(\frac{1}{\sqrt{2}\,\lambda_G^{-1}}\right)\right)} - 1}\right) = \gamma\left(\frac{1}{\tau^{-1}\left(\frac{1}{\sqrt{2}\,\lambda_G^{-1}}\right)}\right).$$

3.22. Theorem. [**Se**, Theorem 3.4] *The inequalities $j_G \leqslant \delta_G \leqslant 2j_G$ hold for every open set $G \subset \mathbb{R}^n$.*

So, we deduce that $\zeta_\delta^j(t) = t$ and $\zeta_j^\delta(t) = 2t$.

3.23. Theorem. [**Se**, Theorem 4.2] *Let $G \subset \mathbb{R}^n$ be a convex domain, then $j_G \leqslant \alpha_G$.*

This means that $\zeta_\alpha^j(t) = t$ for convex domains.

3.24. Theorem. [**Se**, Theorem 6.2] *Let G be a domain in $\mathbb{R}^n$, for which* $\operatorname{card} \partial G \geqslant 2$ *and ∂G is connected. Then, for distinct points $x, y \in G$,*

$$\mu_G(x,y) \geqslant \tau_n\left(\frac{1}{e^{\delta_G(x,y)} - 1}\right).$$

Solving for μ and using a fact that τ_n is decreasing we get:

$$\tau_n^{-1}(\mu_G(x,y)) \leqslant \frac{1}{e^{\delta_G(x,y)-1}}$$

and from here

$$\delta_G(x,y) \leqslant \log\left(1+\frac{1}{\tau_n^{-1}(\mu_G(x,y))}\right).$$

Hence, $\zeta_\mu^\delta(t) = \log\left(1+\frac{1}{\tau_n^{-1}(t)}\right)$ if ∂G is connected and has at least two points.

3.25. THEOREM. [**Se**, Theorem 6.5] *Let $G \subset \overline{\mathbb{R}^n}$ be a domain with card $\partial G \geqslant 2$. Then*

$$\lambda_G(x,y) \leqslant \tau_n\left(\frac{m_G(x,y)}{2}\right).$$

Expressing μ_G in terms of δ_G we get:

$$\lambda_G(x,y) \leqslant \tau_n\left(\frac{e^{\delta_G}-1}{2}\right)$$

and from here we obtain

$$\delta_G(x,y) \leqslant \log\left(1+2\tau_n^{-1}\left(\frac{1}{\lambda_G^{-1}(x,y)}\right)\right).$$

This means that $\zeta_{\lambda^{-1}}^\delta(t) = \log\left(1+2\tau_n^{-1}\left(\frac{1}{t}\right)\right)$ for domains with $card(\partial G) \geq 2$.

At first, we give a 4×4 chart.

	j_G	k_G	μ_G	λ_G^{-1}
j_G	1 $\zeta_j^j(t)=t$	2 $\zeta_j^k(t)=ct$ G – uniform $\zeta_j^k(t)=\varphi(t)$ G – φ domain	3 $\zeta_j^\mu(t) = \gamma\left(\frac{1}{e^t-1}\right)$ locally	4 $\zeta_j^{\lambda^{-1}}(t) = \frac{1}{c\tau(e^{2t}-1)}$ G – c-QED domain
k_G	5 $\zeta_k^j(t)=t$	6 $\zeta_k^k(t)=t$	7 $\zeta_k^\mu(t)=h_2(3t)$	8 $\zeta_k^{\lambda^{-1}} = \zeta_j^{\lambda^{-1}}$
μ_G	9 $\zeta_\mu^j(t) = \frac{1}{c_n}\cdot t$ ∂G connected	10 $\zeta_\mu^k(t) = c\cdot t$ G uniform ∂G connected	11 $\zeta_\mu^\mu(t)=t$	12 $\zeta_\mu^{\lambda^{-1}} = \zeta_\mu^j \circ \zeta_j^{\lambda^{-1}}$ G – c-QED domain ∂G connected
λ_G^{-1}	13 $\zeta_{\lambda^{-1}}^j(t) = \log\left(1+\tau^{-1}\left(\frac{1}{\sqrt{2}t}\right)\right)$	14 $\zeta_{\lambda^{-1}}^k = \zeta_{\lambda^{-1}}^j \circ \zeta_j^k$ G uniform	15 $\zeta_{\lambda^{-1}}^\mu = \zeta_{\lambda^{-1}}^j \circ \zeta_j^\mu$ locally	16 $\zeta_{\lambda^{-1}}^{\lambda^{-1}}(t)=t$

Function ζ_j^μ can be written in a different form using the estimate of γ function. We define functions Φ and Ψ as in [**Vu2**, 7.19] by

$$\gamma_n(s) = \omega_{n-1}(\log(\Phi(s)))^{n-1}, \quad s > 1 \tag{3.26}$$

$$\tau_n(t) = \omega_{n-1}(\log(\Psi(t)))^{n-1}, \quad t > 0. \tag{3.27}$$

3.28. Lemma. [**Vu2**, Lemma 7.22] *For each $n \geqslant 2$ there exists a number $\lambda_n \in [4, 2\,e^{n-1})$, $\lambda_2 = 4$, such that*

$$t \leqslant \Phi(t) \leqslant \lambda_n t, \quad t > 1 \tag{3.29}$$

$$t + 1 \leqslant \Psi(t) \leqslant \lambda_n^2(t+1), \quad t > 0. \tag{3.30}$$

From (3.27) we have that $\omega_{n-1}(\log(\lambda_n^2(t+1)))^{1-n} \leqslant \tau_n(t) \leqslant \omega_{n-1}(\log(t+1))^{1-n}$.
From (3.26) we have

$$\omega_{n-1}\left(\log \lambda_n t\right)^{1-n} \leqslant \gamma_n(t) \leqslant \omega_{n-1}\left(\log t\right)^{1-n}, \quad t > 1.$$

Using the right side of this inequality we have

$$\gamma\left(\frac{1}{e^t - 1}\right) \leqslant \omega_{n-1}\left(\log\left(\frac{1}{e^t-1}\right)\right)^{1-n} \leqslant \omega_{n-1}\left(\log\left(\frac{1}{t}\right)\right)^{1-n}.$$

This gives $\zeta_j^\mu(t) \leqslant \omega_{n-1}\left(\log\left(\frac{1}{t}\right)\right)^{1-n}$ locally.

4. Inclusion relations for balls

Each statement on modulus of continuity has its counterpart stated in terms of inclusions of balls. Namely, if for some metrics d_1 and d_2 holds

$$d_1(x,y) < t \Rightarrow d_2(x,y) < \zeta(t),$$

then

$$D_{d_1}(x,t) \subset D_{d_2}(x, \zeta(t)).$$

A related question is to find, for a given $x \in G$ and $t > 0$, minimal $\zeta(x,t)$ such that

$$D_{d_1}(x,t) \subset D_{d_2}(x, \zeta(x,t)),$$

This is circumscribed ball problem for a fixed $x \in G$.

The quasihyperbolic ball $D_k(x,r)$ is the set $\{z \in G \mid k_G(x,z) < r\}$, when $x \in G$ and $r > 0$. By [**Vu2**, (3.9)], we have the inclusions

$$B^n(x, r\,d(x)) \subset D_k(x,M) \subset B^n(x, R\,d(x)), \tag{4.1}$$

where $r = 1 - e^{-M}$ and $R = e^M - 1$.

It was proved in [**AVV**, 15.13] that if G is a proper subdomain of R^n and if $x, y \in G$ with $x \neq y$, then

$$\lambda_G(x,y) \leqslant \inf_{z \in \partial G}(\lambda_{\mathbb{R}^n \setminus \{z\}}(x,y)) \leqslant \sqrt{2}\tau_n\left(\frac{|x-y|}{\min\{d(x), d(y)\}}\right) \tag{4.2}$$

4.3. THEOREM. [**H**, Theorem 6.11] *Let G be a proper subdomain of $\mathbb{R}^n$ and let $t > 0$. We denote* $c_1 = \frac{1}{(1+\tau_n^{-1}(t/\sqrt{2}))}$, $c_2 = \sqrt{\frac{\tau_n^{-1}(2t)}{(1+\tau_n^{-1}(2t))}}$ *and* $c_3 = \tau_n^{-1}(t/\sqrt{2})$, *then the inclusions*

$$D_{\lambda^{-1}}(a,t) \subset \{z \in G \,|\, d(z) > c_1 d(a)\}, \tag{4.4}$$

$$D_{\lambda^{-1}}(a,t) \supset B^n(a, c_2 d(a)) \supset D_k(a, \log(c_2+1)) \tag{4.5}$$

and

$$D_{\lambda^{-1}}(a,t) \subset B^n(a, c_3 d(a)) \cap G \tag{4.6}$$

are valid for all $a \in G$. If, in addition, $t > \sqrt{2}\tau_n(1)$, we have that

$$B^n(a, c_3 d(a)) \subset D_k(a, \log(1/(1-c_3))). \tag{4.7}$$

To prove the inclusion (4.6), we apply (4.2) to obtain

$$\lambda_G(a,z) \leqslant \sqrt{2}\tau_n\left(\frac{|z-a|}{d(a)}\right).$$

From here with the assumption $t \leqslant \lambda_G(a,z)$ we have $|z-a| < \tau_n^{-1}(t/\sqrt{2})d(a)$.

Since $D_{\lambda^{-1}} \subset G$, the inclusion (4.6) holds.

Inclusion (4.7) follows directly from (4.1) after we notice that the condition $t > \sqrt{2}\tau_n(1)$ implies that $c_3 < 1$ and hence that the ball $B^n(a, c_3 d(a))$ is included in G.

4.8. THEOREM. [**H**, Theorem 6.18] *Let G be a proper subdomain od $\mathbb{R}^n$ and assume that G has a connected, nondegenerate boundary. Let $t > 0$ and denote $d_1 = \tau_n^{-1}(t)/(1+\tau_n^{-1}(t))$, $d_2 = 1/\gamma_n^{-1}(t)$ and $d_3 = 1/\tau_n^{-1}(t)$. Then, for all $a \in G$, the following inclusions hold*

$$D_\mu(a,t) \subset \{z \in G \,|\, d(z) > d_1 d(a)\}, \tag{4.9}$$

$$D_\mu(a,t) \supset B^n(a, d_2 d(a)) \supset D_k(a, \log(d_2+1)) \tag{4.10}$$

$$D_\mu(a,t) \subset B^n(a, d_3 d(a)) \cap G. \tag{4.11}$$

If in addition $t < \tau_n(1)$, then

$$B^n(a, d_3 d(a)) \subset D_k(a, \log(1/(1-d_3))). \tag{4.12}$$

The numbers d_1, d_2 and d_3 are best possible for these inclusions.

We prove (4.10) only, because that part is used later on.

We assume that $a, z \in G$ and that $|z-a| \leqslant d_2\, d(a)$. Then, since $\gamma_n^{-1}(t) > 1$, we have $d(z,a) < d(a)$. We consider the following curve families.

$$\Gamma_J = \Delta(J_{az}, \partial G; G),$$

$$\Gamma = \Delta(J_{az}, S^{n-1}(a, d(a)); \overline{B^n(a, d(a))}),$$

and

$$\tilde{\Gamma} = \Delta([z', +\infty), S^{n-1}; \mathbb{R}^n \setminus B^n), \tag{4.13}$$

where $z' = \frac{d(a)}{|z-a|}\, e_1$. Since J_{az} is a continuum which joins a and z, we have

$$\mu_G(a,z) \leqslant M(\Gamma_J) \tag{4.14}$$

and since $\Gamma < \Gamma_J$, we have that $M(\Gamma_J) < M(\Gamma)$.

Using Möbius transformations, we get

$$M(\Gamma) = M(\tilde{\Gamma}) = \gamma_n\left(\frac{d(a)}{|z-a|}\right), \tag{4.15}$$

and since $|z-a| < d_2\, d(a)$ and γ_n is a strictly decreasing homeomorphism, it follows that

$$\gamma_n\left(\frac{d(a)}{|z-a|}\right) < \gamma_n\left(\frac{1}{d_2}\right) = t. \tag{4.16}$$

Combining all these inequalities, we get

$$\mu_G(a,z) < t,$$

which proves the left side of (4.10). The right side inclusion follows from (4.1).

Theorem 4.3 ((4.6) and (4.7)) gives

4.17. Theorem.

$$\lambda^{-1}(a,b) < \frac{1}{t} \Rightarrow k(a,b) < \log\frac{1}{1-\tau_2^{-1}\left(\frac{t}{\sqrt{2}}\right)}, \quad \textit{for } t > \sqrt{2}\tau_2(1)$$

$$\lambda^{-1}(a,b) < s \Rightarrow k(a,b) < \log\frac{1}{1-\tau_2^{-1}\left(\frac{1}{\sqrt{2}s}\right)},$$

$$\zeta^k_{\lambda^{-1}}(s) = \log\frac{1}{1-\tau_2^{-1}\left(\frac{1}{\sqrt{2}s}\right)}, \quad s < \frac{1}{\sqrt{2}\tau_2(1)}.$$

Also we obtain $\lambda^{-1}(x,a) < \frac{1}{t} \Rightarrow |x-a| < c_3 d(a) < \operatorname{diam}(G)\, c_3(1/t)$ *and from here* $\zeta^{|\cdot|}_{\lambda^{-1}}(t) = \tau_n^{-1}(1/(\sqrt{2}t))\operatorname{diam}(G)$.

From Theorem 4.8 we deduce

4.18. Theorem. *In a domain G with connected nondegenerate boundary:*

$$D_\mu(a,t) \supset D_k(a, \log(d_2+1)), \quad d_2 = \frac{1}{\gamma^{-1}(t)}, \tag{4.19}$$

and $\mu(a,b) < t$ *if* $k(a,b) < \log(d_2+1)$.

Also, $\zeta^\mu_k(s) = \gamma(1/(e^s-1))$. *If we put*

$$s = \log\left(\frac{1}{\gamma^{-1}(t)} + 1\right), \quad \textit{we have } e^s - 1 = \frac{1}{\gamma^{-1}(t)}, \quad t = \gamma\left(\frac{1}{e^s-1}\right).$$

4.20. Theorem. [**Se**, Theorem 3.8] *If $G \subset \mathbb{R}^n$ is open, $x \in G$ and $t > 0$ then*

$$D_j(x,t) \subset B^n(x,R)$$

where $R = (e^t - 1)\, d(x)$. This formula for R is the best possible expressed in terms of t and $d(x)$ only.

Therefore, using $d(x) \leqslant \operatorname{diam}(G)$, we get $\zeta_j^{|\cdot|}(t) = (e^t - 1)\operatorname{diam}(G)$.

4.21. Theorem. [**Se**, Theorem 3.10] *If $G \subset \mathbb{R}^n$ is an open set, $x \in G$ and $t > 0$ then $D_\delta(x,t) \subset B^n(x,R)$ where $R = (e^t - 1)\, d(x)$.*

As above, we get $\zeta_\delta^{|\cdot|}(t) = (e^t - 1)\operatorname{diam}(G)$.

From Lemma 3.7, we have that

$$\zeta_k^\mu(t) = h(3t).$$

Now, from [**H**, Lemma 2.30] we may choose (the case $n = 2$)

$$h(t) = \frac{2\pi\alpha}{\log\frac{1}{2t}}, \quad \text{for } t \leqslant \frac{1}{4}.$$

From here we have that

$$\zeta_k^\mu(t) = \frac{2\pi\alpha}{\log\left(\frac{1}{6t}\right)}, \text{for} \quad t \leqslant \frac{1}{12} \quad \text{(more important case)}$$

$$\alpha = \max\{1,\gamma\}, \quad \gamma = \frac{9}{8}\log 2 > 1\,, \quad \alpha = \gamma\,.$$

In the second case, where

$$h(t) = 36\beta\pi t^2, \quad \text{for } t > \frac{1}{4}.$$

we have $h(3t) = 324\beta\pi t^2$, $t > \frac{1}{12}$.

$$\beta = \max\left(1, \frac{1}{\gamma}\right) = 1$$

$$\zeta_k^\mu(t) = 324\pi t^2, \quad \text{for } t > \frac{1}{12}.$$

	j_G	k_G	μ_G	λ_G^{-1}
j_G	1 $\zeta_j^j(t)=t$	2 $\zeta_j^k(t)=ct$ G – uniform $\zeta_j^k(t)=\varphi(t)$ $G-\varphi$ domain	3 $\zeta_j^\mu(t) = \omega_{n-1}\left(\log\left(\frac{1}{t}\right)\right)^{1-n}$ locally	4 $\zeta_j^{\lambda^{-1}}(t) = \dfrac{1}{c\tau(e^{2t}-1)}$ G – c-QED domain
k_G	5 $\zeta_k^j(t)=t$	6 $\zeta_k^k(t)=t$	7 $\zeta_k^\mu(t) = \gamma\left(\frac{1}{e^t-1}\right)$ ∂G connected, nondegenerate	8 $\zeta_k^{\lambda^{-1}}=\zeta_j^{\lambda^{-1}}$
μ_G	9 $\zeta_\mu^j(t)=\frac{t}{c_n}$ ∂G connected	10 $\zeta_\mu^k(t)=c\cdot t$ G uniform ∂G connected	11 $\zeta_\mu^\mu(t)=t$	12 $\zeta_\mu^{\lambda^{-1}}=\zeta_\mu^j\circ\zeta_j^{\lambda^{-1}}$ G – c-QED domain ∂G connected
λ_G^{-1}	13 $\zeta_{\lambda^{-1}}^j(t) = \log\left(1+\tau^{-1}\left(\frac{1}{\sqrt{2}\,t}\right)\right)$	14 $\zeta_{\lambda^{-1}}^k(t) = \log\frac{1}{1-\tau_2^{-1}(1/(\sqrt{2}t))}$ $t<\frac{1}{\sqrt{2}\tau_2(1)}$	15 $\zeta_{\lambda^{-1}}^\mu=\zeta_{\lambda^{-1}}^j\circ\zeta_j^\mu$ locally	16 $\zeta_{\lambda^{-1}}^{\lambda^{-1}}(t)=t$

This is improved 4×4 chart.

4.22. EXAMPLE. *For $G\subset\mathbb{R}^n$ we choose $z_0\in\partial G$, sequence $x_k\in G$ such $x_k\to z_0$ and sequence $y_k\in G$ such that*

$$|y_k-z_0|<\frac{|x_k-z_0|}{k}. \tag{4.23}$$

Clearly $|x_k-y_k|\to 0$ and

$$|x_k-y_k|>|x_k-z_0|-|y_k-z_0|>|x_k-z_0|\left(1-\frac{1}{k}\right). \tag{4.24}$$

But

$$j_G(x_k,y_k)\geqslant\log\left(1+\frac{|x_k-y_k|}{|y_k-z_0|}\right)\geqslant\log\left(1+\frac{1-\frac{1}{k}}{\frac{1}{k}}\right)=\log(k)\to+\infty.$$

Hence $id:(G,|\cdot|)\longrightarrow(G,j_G)$ is not uniformly continuous. By this reason, adequate fields in the chart are empty.

Also, for a fixed small $d>0$ we can find $x,y\in G$ such that $|x-y|=d$ and $d(x,\partial G)$ as small as we like.

So we get $k_G(x,y)$ as large as we like and there is no estimate of $k_G(x,y)$ in terms of $|x-y|$.

In other hand function $\zeta_k^{|\cdot|}$ is obtained from:

$$k_G(x,y) \geqslant \int_0^{|x-y|} \frac{ds}{\operatorname{diam}(G)} = \frac{|x-y|}{\operatorname{diam}(G)}.$$

From here we get that modulus of continuity of $id : (G, k_G) \longrightarrow (G, |\cdot|)$ is $\zeta_k^{|\cdot|}(t) = t\operatorname{diam}(G)$ (where G is bounded).

All the remaining items are obtained by composition of the above moduli of continuity.

And finally we have following charts:

	α_G	δ_G	j_G	k_G
α_G	$\zeta_\alpha^\alpha(t) = t$	$\zeta_\alpha^\delta(t) = 2t$ G convex	$\zeta_\alpha^j(t) = t$ G convex	$\zeta_\alpha^k(t) = ct$ G convex, uniform
δ_G	$\zeta_\delta^\alpha(t) = t$	$\zeta_\delta^\delta(t) = t$	$\zeta_\delta^j(t) = t$	$\zeta_\delta^k(t) = ct$ G uniform
j_G	$\zeta_j^\alpha(t) = 2t$	$\zeta_j^\delta(t) = 2t$	$\zeta_j^j(t) = t$	$\zeta_j^k(t) = ct$ G uniform
k_G	$\zeta_k^\alpha(t) = 2t$	$\zeta_k^\delta(t) = 2t$	$\zeta_k^j(t) = t$	$\zeta_k^k(t) = t$

	q	$\lvert\cdot\rvert$	μ_G	λ_G^{-1}
α_G	$\zeta_\alpha^q(t) = (e^t - 1)\operatorname{diam}(G)$ G convex	$\zeta_\alpha^{\lvert\cdot\rvert}(t) = (e^t - 1)\operatorname{diam}(G)$ G convex	$\zeta_\alpha^\mu(t) = \gamma\left(\dfrac{1}{e^t-1}\right)$ G convex, locally	$\zeta_\alpha^{\lambda^{-1}}(t) = \dfrac{1}{c\tau(e^{2t}-1)}$ G c-QED, convex
δ_G	$\zeta_\delta^q(t) = (e^t - 1)\operatorname{diam}(G)$	$\zeta_\delta^{\lvert\cdot\rvert}(t) = (e^t - 1)\operatorname{diam}(G)$	$\zeta_\delta^\mu(t) = \gamma\left(\dfrac{1}{e^t-1}\right)$ locally	$\zeta_\delta^{\lambda^{-1}}(t) = \dfrac{1}{c\tau(e^{2t}-1)}$ G c-QED
j_G	$\zeta_j^q(t) = (e^t - 1)\operatorname{diam}(G)$	$\zeta_j^{\lvert\cdot\rvert}(t) = (e^t - 1)\operatorname{diam}(G)$	$\zeta_j^\mu(t) = \gamma\left(\dfrac{1}{e^t-1}\right)$ locally	$\zeta_j^{\lambda^{-1}}(t) = \dfrac{1}{c\tau(e^{2t}-1)}$ G c-QED
k_G	$\zeta_k^q(t) = t\operatorname{diam}(G)$	$\zeta_k^{\lvert\cdot\rvert}(t) = t\operatorname{diam}(G)$	$\zeta_k^\mu(t) = \gamma\left(\dfrac{1}{e^t-1}\right)$ ∂G connected nondegenerate	$\zeta_k^{\lambda^{-1}}(t) = \dfrac{1}{c\tau(e^{2t}-1)}$ G c-QED

	α_G	δ_G	j_G	k_G
q	Does not exist	Does not exist	Does not exist	Does not exist
$\lvert\cdot\rvert$	Does not exist	Does not exist	Does not exist	Does not exist
μ_G	$\zeta_\mu^\alpha(t) = \log\left(1 + \frac{1}{\tau^{-1}(t)}\right)$ ∂G connected	$\zeta_\mu^\delta(t) = \log\left(1 + \frac{1}{\tau^{-1}(t)}\right)$ ∂G connected $card(\partial G) \geq 2$	$\zeta_\mu^j(t) = \frac{t}{c_n}$ ∂G connected	$\zeta_\mu^k(t) = ct$ G uniform ∂G connected
λ_G^{-1}	$\zeta_{\lambda^{-1}}^\alpha(t) = \log\left(1 + 2\tau^{-1}(\frac{1}{t})\right)$	$\zeta_{\lambda^{-1}}^\delta(t) =$ $\log\left(1 + 2\tau^{-1}(\frac{1}{t})\right)$ $card(\partial G) \geq 2$	$\zeta_{\lambda^{-1}}^j(t) = \log(1 + \tau^{-1}(\frac{1}{\sqrt{2}t}))$	$\zeta_{\lambda^{-1}}^k(t) = c\log(1+\tau^{-1}(\frac{1}{\sqrt{2}t}))$ G uniform

	q	$\lvert\cdot\rvert$	μ_G	λ_G^{-1}
q	$\zeta_q^q(t) = t$	$\zeta_q^{\lvert\cdot\rvert}(t) = ct$ G bounded	Does not exist	Does not exist
$\lvert\cdot\rvert$	$\zeta_{\lvert\cdot\rvert}^q(t) = t$	$\zeta_{\lvert\cdot\rvert}^{\lvert\cdot\rvert}(t) = t$	Does not exist	Does not exist
μ_G	$\zeta_\mu^q(t) = \frac{\mathrm{diam}(G)}{\tau^{-1}(t)}$ ∂G connected	$\zeta_\mu^{\lvert\cdot\rvert}(t) = \frac{\mathrm{diam}(G)}{\tau^{-1}(t)}$ ∂G connected	$\zeta_\mu^\mu(t) = t$	$\zeta_\mu^{\lambda^{-1}} = \frac{1}{c\tau(e^{bt}-1)}$ G c-QED domain ∂G connected
λ_G^{-1}	$\zeta_{\lambda^{-1}}^q = \tau^{-1}(1/(\sqrt{2}t))\,\mathrm{diam}(G))$	$\zeta_{\lambda^{-1}}^{\lvert\cdot\rvert} = \tau^{-1}(1/(\sqrt{2}t))\,\mathrm{diam}(G))$	$\zeta_{\lambda^{-1}}^\mu = \gamma\left(\frac{1}{\tau^{-1}\left(\frac{1}{\sqrt{2}t}\right)}\right)$ locally	$\zeta_{\lambda^{-1}}^{\lambda^{-1}}(t) = t$

Sharper results can be obtained for special domains, for example $G = \mathbb{R}^n \setminus \{0\}$ was studied by R. Klen **[Kl]** in relation to j_G metrics.

We return to the question of moduli of continuity, from a different viewpoint, in chapter 2, sections 2 and 3.

5. Removing a point

Let $\mathcal{M}$ be a collection of metrics on a domain $G \subset \mathbb{R}^n$ and $B_m(x,T) = \{z \in G : m(x,z) < T\}$, $m \in \mathcal{M}$. Let

$$r_T = \sup\{r > 0 \,:\, S^{n-1}(x,r) \subset B_m(x,T)\},$$
$$R_T = \inf\{r > 0 \,:\, S^{n-1}(x,r) \cap B_m(x,T) = \emptyset\}.$$

The question is can we find lower bound for r_T and upper bound for R_T.

5.1. Problem. (Radius of circumscribed ball)

It is evident from the definition of λ_G that adding new points, even isolated ones, to the boundary of G will affect the value of $\lambda_G(x,y)$ for fixed points $x, y \in G$. We study this phenomenon in the case when $G = \mathbb{R}^2 \setminus \{0\}$.

We find an upper bound for radius of circumscribed ball, where $m = \lambda_G^{-1}$.

We use notation

$$B_\lambda(1,T) = \{z \in \mathbb{C} \,:\, \lambda_G(z,1) \geqslant T^{-1}\}.$$

Let $h(z) = \frac{z}{|z|^2}$ be an inversion. Since $h : B_\lambda \longrightarrow B_\lambda$ (h is an isometry for λ metric) we have

$$\lambda_G(1,z) = \lambda_G(1,h(z)).$$

From **[SolV**, (3.3), (3.22)**]** we have

$$p(z) = \frac{2\pi}{\log M(2z-1)}, \quad z \in \mathbb{C} \setminus \{0,1\} \quad \text{and} \tag{5.2}$$

$$\log M(2e^{i\theta}-1) = \frac{2\pi\, \mathcal{K}(\sin\frac{\theta}{4})\, \mathcal{K}(\cos\frac{\theta}{4})}{\mathcal{K}^2(\sin\frac{\theta}{4}) + \mathcal{K}^2(\cos\frac{\theta}{4})}. \tag{5.3}$$

If we put $z = e^{i\theta}$ we have

$$p(e^{i\theta}) = \frac{\mathcal{K}^2(\sin\frac{\theta}{4}) + \mathcal{K}^2(\cos\frac{\theta}{4})}{\mathcal{K}(\sin\frac{\theta}{4})\, \mathcal{K}(\cos\frac{\theta}{4})}.$$

For $|z| = 1$ we obtain $\lambda_G(1,z) = p(z)$.

Choose θ such that $\sin\frac{\theta}{2} = \frac{R_T}{2}$. From here $\theta = 2\arcsin\frac{R_T}{2}$. Now if we put

$$y = \frac{\mathcal{K}(\sin\frac{\theta}{4})}{\mathcal{K}(\cos\frac{\theta}{4})} = \frac{2}{\pi}\mu(\cos\frac{\theta}{4}) \tag{5.4}$$

we have

$$p(e^{i\theta}) = y + \frac{1}{y} = \frac{1}{T}.$$

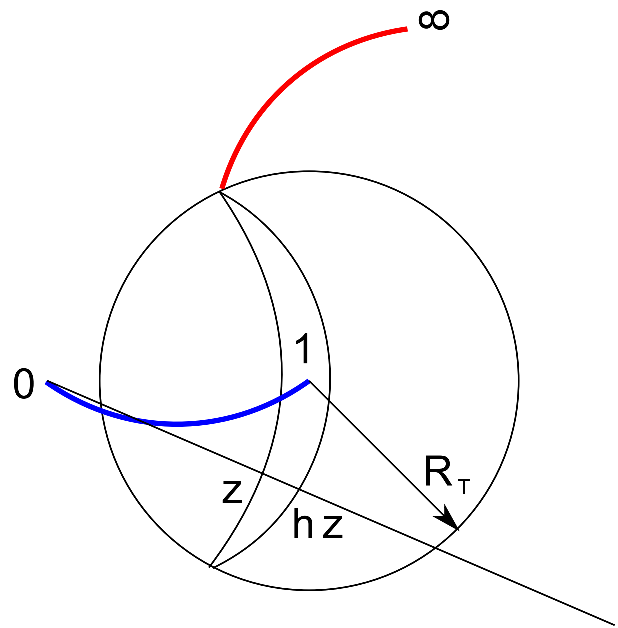

FIGURE 1. Radius of circumscribed ball

We are interested for solutions $y < 1$ because we want $\theta < \pi$. From here $y = \frac{2T}{1+\sqrt{1-4T^2}}$. Since from (5.4)

$$\theta = 4\arccos(\mu^{-1}(\frac{\pi y}{2}))$$

now we have

$$\theta = 4\arccos(\mu^{-1}(\frac{\pi}{2}\frac{2T}{1+\sqrt{1-4T^2}})) = 4\arccos(\mu^{-1}(\frac{\pi T}{1+\sqrt{1-4T^2}})). \tag{5.5}$$

Hence, the radius of the circumscribed sphere is

$$R_T = 2\sin\frac{\theta}{2}, \quad T \in (0, \frac{1}{2}), \quad \theta \text{ from } (5.5).$$

5.6. OPEN QUESTION. (1) Can we find r_T in the case above?

(2) Can we estimate R_T, where G is now bounded subset of $\mathbb{C}$ (instead of $\mathbb{R}^2 \setminus \{0\}$)?
(3) Consider μ_G-balls where ∂G is connected, say $\partial G = [0, e_1]$. Can we find a lower bound for r_T (upper bound for R_T) in this case?

5.7. PROBLEM. (Estimate for $\lambda_{B^2\setminus\{0\}}(x,y)$) Next we investigate the following situation: $G \subseteq \mathbb{R}^n$ is domain, $a \in G$, $G' = G \setminus \{a\}$. Is $\lambda_G(x,y) = \lambda_{G'}(x,y)$ true under some additional assumptions, like x, y close to ∂G?

We consider a special case where $G = B^2$ and $a = 0$.

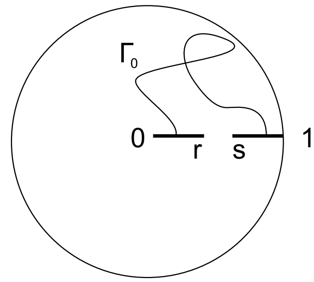

FIGURE 2.

In [**LeVu**, Lemma 2.8] is proven that if $\Gamma_0 = \Delta([0,x],[\tilde{y},x/|x|];B)$, where $\tilde{y} = \frac{|y|}{|x|}x$ and if we put $|x| = r$, $|\tilde{y}| = s$, then we have

$$M(\Gamma_0) = \tau\left(\frac{(s-r)(1-rs)}{r(1-s)^2}\right). \tag{5.8}$$

Further, from [**Vu1**, (2.6)] we have that if $\Delta_0 = \Delta([-\frac{x}{|x|},-x],[x,\frac{x}{|x|}];B)$ and if $|x| = r$ as before, then

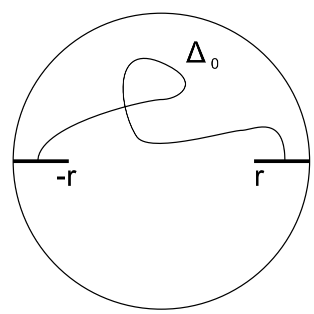

FIGURE 3.

$$M(\Delta_0) = \frac{1}{2}\tau\left(\frac{4r^2}{(1-r^2)^2}\right).$$

Also, using Möbius transformation $T_r : B^2 \longrightarrow B^2$, $T(r) = 0$ we can map family of curves Δ_1 to family of curves Δ_1', where $\Delta_1 = \Delta([-\frac{x}{|x|},-\tilde{y}],[0,x];B)$ and $\Delta_1' = \Delta([-\frac{x}{|x|},-\tilde{y}'],[-x,0];B)$.

We know that

$$\rho(-s,0) = \rho(-r,-t),$$

where r and s are as before and $-t = T_r(-s)$. Further, this is equivalent to

$$\log\frac{1+s}{1-s} = \log\frac{1+t}{1-t}\frac{1-r}{1+r}. \tag{5.9}$$

Solving (5.9) in t we obtain $t = \frac{s+r}{1+sr}$.

Now we have

$$M(\Delta_1) = M(\Delta_1') = \tau\left(\frac{(t-r)(1-tr)}{r(1-t)^2}\right) = \tau\left(\frac{s(1+r)^2}{r(1-s)^2}\right).$$

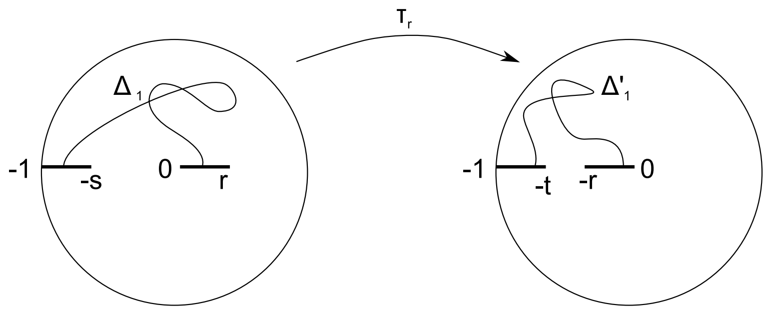

FIGURE 4.

The first equality holds because T_r is conformal map, the second one follows from (5.8) and the third one from the expression for t.

Now, if we put in last term that $r = s$, we obtain

$$M(\Delta_1) = \tau\left(\left(\frac{1+r}{1-r}\right)^2\right).$$

The question is when is $M(\Delta_1) \geqslant M(\Delta_0)$. In other words, when is

$$\tau\left(\left(\frac{1+r}{1-r}\right)^2\right) \geqslant \frac{1}{2}\tau\left(\frac{4r^2}{(1-r^2)^2}\right)? \tag{5.10}$$

Applying formula [**AVV**, 5,19 (5)]:

$$\frac{1}{2}\tau(t) \geqslant \tau((\sqrt{t}+\sqrt{t+1})^4 - 1)$$

for $t = 4r^2/(1-r^2)^2$ we have

$$\frac{1}{2}\tau\left(\frac{4r^2}{(1-r^2)^2}\right) = \tau\left(\frac{8r(r^2+1)}{(1-r)^4}\right).$$

Then (5.10) is equivalent to

$$\left(\frac{1+r}{1-r}\right)^2 \leqslant \frac{8r(r^2+1)}{(1-r)^4},$$

since τ is decreasing. The last inequality is equivalent to

$$r^4 - 8r^3 - 2r^2 - 8r + 1 \leqslant 0.$$

This inequality holds for $r \in [0.12, 1)$.

This gives the answer to the question: For which values of $|x|$ we have

$$\lambda_A(x, -x) = M(\Delta(E, -E; B^2)),$$

where $A = B^2 \setminus \{0\}$, $E = [x, \frac{x}{|x|}]$?

A related result can be found in Heikkala's dissertation, [**H**, Theorem 7.3]. In fact, this theorem deals with the more general situation: If x and y are close to the boundary and far apart then $\lambda_{B^n\setminus\{0\}}(x,y) = \lambda_{B^n}(x,y)$. His theorem is:

5.11. THEOREM. *Let $G = B^n \setminus \{0\}$ and let $x, y \in G$ with $|x-y| \geq \delta > 0$. Then, if $\min\{|x|,|y|\} \in (r_1, 1)$ with $r_1 = \frac{\sqrt{\delta^4+64}-\delta^2}{8}$, we have that*

$$\lambda_G(x,y) = \lambda_{B^n}(x,y).$$

However, we have in the special case $x = -y$, better constant (letting $\delta = 2|x|$ and $r_1 = |x|$ in Theorem 7.3 gives equation $r_1^3 + r_1^2 - 1 = 0$, and its real root is larger than 0.75, and consequently larger than 0.12).

6. Uniform continuity on union of two domains

6.1. DEFINITION. *Let $\{m_D : D \subseteq \overline{\mathbb{R}^n}\}$ be a family of metrics. We say that this family is monotone if $D_1 \subseteq D_2$ implies $m_{D_1}(x,y) \geq m_{D_2}(x,y)$ for all $x, y \in D_1$.*

6.2. LEMMA. [**Vu5**, 2.27] *Let G_1, G_2 be domains in $\mathbb{R}^n$ with $G_1 \cap G_2 \neq \emptyset$, $G_1 \neq \mathbb{R}^n \neq G_2$ and assume that there exists $c \in (0,1)$ such that*

$$d(x, \partial G_1) + d(x, \partial G_2) \geq c\, d(x, \partial(G_1 \cup G_2)), \tag{6.3}$$

for all $x \in G = G_1 \cup G_2$.

Suppose that $f : G \longrightarrow fG$ is continuous, $fG \subseteq \mathbb{R}^n$; that $\{m_D : D \subseteq \mathbb{R}^n\}$ is a monotone family of metrics; and that

$$m_{fG_j}(f(x), f(y)) \leq \omega_j(k_{G_j}(x,y)) \tag{6.4}$$

for $x, y \in G_j$ and $j = 1, 2$. Then there exists $\omega : [0, +\infty) \longrightarrow [0, +\infty)$ such that

$$m_{fG}(f(x), f(y)) \leq \omega(k_G(x,y)) \tag{6.5}$$

and $\lim_{t\to 0+} \omega(t) = 0$ provided $\lim_{t\to 0+} \omega_j(t) = 0$, $j = 1, 2$.

Now we consider a similar, but local result, with j metric replacing k metric. We can no longer use geodesics as was done in the proof of the above lemma.

6.6. LEMMA. *Let G_1, G_2 be domains in $\mathbb{R}^n$ with $G_1 \cap G_2 \neq \emptyset$, $G_1 \neq \mathbb{R}^n \neq G_2$ and assume that there exists $c \in (0,1)$ such that*

$$d(x, \partial G_1) + d(x, \partial G_2) \geq c\, d(x, \partial(G_1 \cup G_2)),$$

for all $x \in G = G_1 \cup G_2$.

Suppose that $f : G \longrightarrow fG$ is continuous, $fG \subseteq \mathbb{R}^n$; that $\{m_D : D \subseteq \mathbb{R}^n\}$ is a monotone family of metrics; and that

$$m_{fG_j}(f(x), f(y)) \leq \omega_j(j_{G_j}(x,y))$$

for $x, y \in G_j$ and $j = 1, 2$. Then there exists $\omega : [0, \delta) \longrightarrow [0, +\infty)$, where $\delta = \log\left(1 + \frac{c}{4}\right)$ such that

$$m_{fG}(f(x), f(y)) \leq \omega(j_G(x,y)) \tag{6.7}$$

for $x, y \in G$, $j_G(x,y) \le \delta$ *and* $\lim_{t\to 0+} \omega(t) = 0$ *provided* $\lim_{t\to 0+} \omega_j(t) = 0$, $j = 1, 2$.

Proof. Let $d(x) = d(x, \partial G)$ and $j_G(x,y) \le \delta$.

Then, we have $|x-y| \le \frac{c}{4} \min\{d(x), d(y)\}$. We may assume $d(x) \le d(y)$. By the hypothesis (6.3) of the lemma there exists $i \in \{1,2\}$ such that $d(x, \partial G_i) \ge \frac{c}{2} d(x)$, i.e., $B^n(x, c\, d(x)/2) \subseteq G_i$. Without loss of generality, we may assume that $i = 1$. Then

$$y \in B^n(x, c \min\{d(x), d(y)\}/4) \subseteq B^n(x, \frac{1}{2} d(x, \partial G_1)).$$

We have

$$j_{G_1}(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d_1(x), d_1(y)\}}\right)$$

where $d_1(z) = d(z, \partial G_1)$. By the above calculation, $|x-y| \le \frac{1}{2} d_1(x)$ and hence $d_1(y) \ge \frac{1}{2} d_1(x)$. The last inequality now yields

$$\begin{aligned} j_{G_1}(x,y) &\le \log\left(1 + \frac{2|x-y|}{d_1(x)}\right) \\ &\le \log\left(1 + \frac{4|x-y|}{c\, d(x)}\right) \\ &< \frac{4}{c} j_G(x,y). \end{aligned}$$

Conclusion:

$$m_{fG}(f(x), f(y)) \le m_{fG_1}(f(x), f(y)) \le \omega_1(j_{G_1}(x,y)) \le \omega_1\left(\frac{4}{c} j_G(x,y)\right) \tag{6.8}$$

where also monotone property of the family $\{m_D\}$ was applied. $\square$

6.9. Example. We present an example, due to J. Ferrand, of two domains $G_1, G_2 \subseteq \mathbb{C}$, $G_1, G_2 \ne \emptyset$ and an analytic function $f : H \longrightarrow \mathbb{C}$, $H = G_1 \cup G_2$ such that

(1) (6.4) holds in G_1 and G_2.
(2) (6.5) does not hold on H.

We set $G_1 = \mathbb{C} \setminus \{p + iq : p, q \in \mathbb{Z}\}$ and $G_2 = \mathbb{C} \setminus (\{0\} \cup \{p + 1/2 + iq : p, q \in \mathbb{Z}\})$.

Note that $G_1 \cap G_2 \ne \emptyset$ and $G_1 \cup G_2 = H = \mathbb{C} \setminus \{0\}$. We define $f(\xi) = e^{4\pi\xi}$. This is an entire function.

It is easy to see that $f(G_1) = f(G_2) = f(H) = H$. In fact $f(\Omega_k) = H$, where $\Omega_k = \{x + iy : k < y < k+1\}$. Quasihyperbolic distance in H satisfies

$$k_H(w_1, w_2) = \inf_{e^{z_1} = w_1, e^{z_2} = w_2} |z_1 - z_2|. \tag{6.10}$$

Also, for $i = 1, 2$ holds $d(\xi, \partial G_i) \leqslant 1/2$, so the metric density $\frac{1}{d(\xi, \partial G_i)}$ exceeds $\sqrt{2}$ and therefore (by a line integration)

$$k_{G_i}(\xi_1, \xi_2) \geqslant \sqrt{2} |\xi_1 - \xi_2|. \tag{6.11}$$

Now, (6.10) tells us that $f : G_i \longrightarrow H$ is Lipschitz with respect to euclidean metric in G_i and quasihyperbolic metric in H, and by (6.11) it is also Lipschitz with respect to quasihyperbolic metric in H and G_i.

But f is not uniformly continuous as a map $(H, k_H) \longrightarrow (H, k_H)$: in fact we have

$$\lim_{n\to\infty} k_H(n, n+1) = \log \frac{n+1}{n} = 0,$$

while

$$\lim_{n\to\infty} k_H(f(n), f(n+1)) = \log \frac{e^{4\pi(n+1)}}{e^{4\pi n}} = 4\pi.$$

Note that our domains fail to meet condition (6.10) from [**Vu5**, Lemma 2.27]. Indeed for large $|x|$ we have

$$d(x, \partial(G_1 \cup G_2)) = |x|$$

and

$$d(x, \partial G_1) + d(x, \partial G_2) \leqslant 2\frac{1}{\sqrt{2}} = \sqrt{2},$$

so there is no $c \in (0, 1)$ such that (6.10) is valid.

6.12. Remark. (1) *It would be of interest to find a homeomorphism f with properties as in Example 6.9 due to Ferrand.*
(2) *It is natural to expect that there is a counterpart of Lemma 6.6 for other metrics in place of j.*
(3) *Is the condition (6.3) invariant under the quasiconformal mappings?*

7. Quasiconformal maps with identity boundary values

For a domain $G \subset \mathbb{R}^n$, $n \geqslant 2$, let

$$Id(\partial G) = \{f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n} \text{ homeomorphism } : f(x) = x, \quad \forall x \in \overline{\mathbb{R}^n} \setminus G\}.$$

Here $\overline{\mathbb{R}^n}$ stands for the Möbius space $\mathbb{R}^n \cup \{\infty\}$. We shall always assume that $card\{\overline{\mathbb{R}^n} \setminus G\} \geq 3$. If $K \geqslant 1$, then the class of K-quasiconformal maps in $Id(\partial G)$ is denoted by $Id_K(\partial G)$. Here we use notation and terminology from Väisälä's book [**V2**]. In particular, K-quasiconformal maps are defined in terms of the maximal dilatation as in [**V2**, p. 42] if not otherwise stated.

We will study the following well-known problem:

7.1. Problem. (1) Given $a, b \in G$ and $f \in Id(\partial G)$ with $f(a) = b$, find a lower bound for $K(f)$.
(2) Given $a, b \in G$, construct $f \in Id(\partial G)$ with $f(a) = b$ and give an upper bound for $K(f)$.

O. Teichmüller studied this problem in the case when G is a plane domain with $card(\overline{\mathbb{R}^2} \setminus G) = 3$ and proved the following theorem with a sharp bound for $K(f)$.

7.2. THEOREM. *Let $G = \mathbb{R}^2 \setminus \{0,1\}$, $a, b \in G$. Then there exists $f \in Id_K(\partial G)$ with $f(a) = b$ iff*

$$\log(K(f)) \geqslant s_G(a,b),$$

where $s_G(a,b)$ is the hyperbolic metric of G.

7.3. THEOREM. *If $f \in Id_K(\partial B^n)$, then for all $x \in B^n$*

$$\rho_{B^n}(f(x),x) \leqslant \log\frac{1-a}{a}, \quad a = \varphi_{1/K,n}(1/\sqrt{2})^2,$$

where $\varphi_{K,n}$ is as in (7.15).

7.4. THEOREM. *If $f \in Id_K(\partial B^n)$, then for all $x \in B^n, n \geq 2$, and $K \in [1,17]$*

$$|f(x) - x| \leq \frac{9}{2}(K-1)\,. \tag{7.5}$$

For $n = 2$ we have

$$|f(x) - x| \leqslant \frac{b}{2}(K-1), \quad b \leqslant 4.38. \tag{7.6}$$

The theory of K-quasiregular mappings in $\mathbb{R}^n, n \geq 3$, with maximal dilatation K close to 1 has been extensively studied by Yu. G. Reshetnyak [**R**] under the name "stability theory". By Liouville's theorem we expect that when $n \geq 3$ is fixed and $K \to 1$ the K-quasiregular maps "stabilize", become more and more like Möbius transformations, and this is the content of the deep main results of [**R**] such as [**R**, p. 286]. We have been unable to decide whether Theorem 7.3 follows from Reshetnyak's stability theory in a simple way. V. I. Semenov [**S**] has also made significant contributions to this theory. For the plane case P. P. Belinskii has found several sharp results in [**Bel**].

7.7. PROBLEM. It seems possible that there is a new kind of stability behavior: If $K > 1$ is fixed, do maps in $Id_K(\partial B^n)$ approach identity when $n \to \infty$? Our results do not answer this question. This kind of behavior is anticipated in [**AVV**, Open problem 9, p. 478].

7.8. LEMMA. *For $x, y \in B^n$ let $t = \sqrt{(1-|x|^2)(1-|y|^2)}$. Then for $x, y \in B^n$*

$$\tanh^2\frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2+t^2}\,, \tag{7.9}$$

$$|x-y| \leqslant 2\tanh\frac{\rho_{B^n}(x,y)}{4} = \frac{2|x-y|}{\sqrt{|x-y|^2+t^2}+t}\,, \tag{7.10}$$

where equality holds for $x = -y$.

Next, we consider a decreasing homeomorphism $\mu : (0,1) \longrightarrow (0,\infty)$ defined by

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}\,, \tag{7.11}$$

where $\mathcal{K}(r)$ is Legendre's complete elliptic integral of the first kind and $r' = \sqrt{1-r^2}$, for all $r \in (0,1)$. The Hersch-Pfluger distortion function is an increasing homeomorphism $\varphi_K : (0,1) \longrightarrow (0,1)$ defined by

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K) \tag{7.12}$$

for all $r \in (0,1)$, $K > 0$. By continuity we set $\varphi_K(0) = 0$, $\varphi_K(1) = 1$. From (7.11) we see that $\mu(r)\mu(r') = \left(\frac{\pi}{2}\right)^2$ and from this we are able to conclude a number of properties of φ_K. For instance, by [**AVV**, Thm 10.5, p. 204]

$$\varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1, \quad r' = \sqrt{1-r^2}, \tag{7.13}$$

holds for all $K > 0$, $r \in (0,1)$.

7.14. Special function $\varphi_{K,n}$ We use the standard notation

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}. \tag{7.15}$$

Then $\varphi_{K,n} : (0,1) \longrightarrow (0,1)$ is an increasing homeomorphism, see [**Vu2**, (7.44)]. Because $\gamma_2(1/r) = 2\pi/\mu(r)$ by [**Vu2**, (5.56)], it follows that $\varphi_{K,2}(r)$ is the same as the $\varphi_K(r)$ in (7.12).

7.16. The key constant. The special functions introduced above will have a crucial role in what follows. For the sake of easy reference we give here some well-known identities between them that can be found in [**AVV**]. First, the function

$$\eta_{K,n}(t) = \tau_n^{-1}(\tau_n(t)/K) = \frac{1-\varphi_{1/K,n}(1/\sqrt{1+t})^2}{\varphi_{1/K,n}(1/\sqrt{1+t})^2},\ K > 0\,, \tag{7.17}$$

defines an increasing homeomorphism $\eta_{K,n} : (0,\infty) \to (0,\infty)$ (cf. [**AVV**, p.193]). The constant $(1-a)/a, a = \varphi_{1/K,n}(1/\sqrt{2})^2$, in (7.5) can be expressed as follows for $K > 1$

$$(1-a)/a = \eta_{K,n}(1) = \tau_n^{-1}(\tau_n(1)/K)\,. \tag{7.18}$$

Furthermore, by (7.13)

$$\eta_{K,2}(t) = \frac{s^2}{1-s^2}, \quad s = \varphi_{K,2}(\sqrt{t/(1+t)}) \tag{7.19}$$

and

$$\eta_{K,2}(1) \in (e^{\pi(K-1)}, e^{b(K-1)}) \tag{7.20}$$

where $b = (4/\pi)\,\mathcal{K}(1/\sqrt{2})^2 = 4.376879...$ Note that the constant $\lambda(K)$ in [**AVV**, 10.33 p. 218.] is the same as $\eta_{K,2}(1)$.

For the proof of Lemma 7.29, we record a lower bound for $\varphi_{1/K,n}(r)$. The constant λ_n is the so called Grötzsch ring constant, see [**AVV**].

7.21. Lemma. ([**Vu2**, 7.47, 7.50]) *For* $n \geq 2, K \geq 1,$ *and* $0 \leq r \leq 1$

$$\varphi_{1/K,n}(r) \geq \lambda_n^{1-\beta} r^\beta, \;\; \beta = K^{1/(n-1)}, \tag{7.22}$$

$$\lambda_n^{1-\beta} \geq 2^{1-\beta} K^{-\beta} \geq 2^{1-K} K^{-K} . \tag{7.23}$$

7.24. Lemma. (1) *For all* $m, n \geqslant 1$ *there is* $M > 1$ *such that the inequality*

$$\log(2^{mx-m+1} x^{nx} - 1) \leqslant (2m \log 2 + 2n)(x-1) \tag{7.25}$$

holds for $x \in [1, M]$ *with equality only for* $x = 1$. *Moreover, with* $t = (m \log 2 - n)/(2n)$, M *can be chosen as*

$$M = \sqrt{\frac{(m-1)\log 2 + \log\left(1 + \frac{(n+m\log 2)^2}{n}\right)}{n} + t^2} - t.$$

(2) *Let* $p(x) = \log(2^{mx-m+1} x^{nx} - 1)$, $q(x) = (2m \log 2 + 2n)(x-1)$ *and let us use the above notation. Let* $a_0 = M$ *and* $a_{n+1} = p^{-1}(q(a_n))$ *for* $n \geqslant 1$. *Then the sequence* a_n *is increasing and bounded. If* $a = \lim_{n\to} a_n$ *then the inequality (7.25) holds for* $x \in [1, a]$ *with equality iff* $x \in \{1, a\}$. *For* $m = 3$ *and* $n = 2$ *we have* $a > 17$.

Proof. Let

$$u(x) = (mx - m + 1)\log 2 + nx \log x, \quad v(x) = \log(e^{u(x)} - 1) = \log(2^{mx-m+1} x^{nx} - 1).$$

Then we have

$$\begin{aligned} v''(x) &= (\log(e^{u(x)} - 1))'' = \left(\frac{u'(x)\, e^{u(x)}}{e^{u(x)} - 1}\right)' \\ &= \frac{(u''(x)e^{u(x)} + (u'(x))^2 e^{u(x)})(e^{u(x)} - 1) - (u'(x)\, e^{u(x)})^2}{(e^{u(x)} - 1)^2} \\ &= \frac{e^{u(x)}}{(e^{u(x)} - 1)^2} \cdot ((u''(x) + (u'(x))^2)(e^{u(x)} - 1) - (u'(x))^2 e^{u(x)}) \\ &= \frac{e^{u(x)}}{(e^{u(x)} - 1)^2} \cdot (u''(x)(e^{u(x)} - 1) - (u'(x))^2). \end{aligned}$$

Thus

$$v''(x) \leqslant 0 \;\; \Leftrightarrow \;\; u''(x)(e^{u(x)} - 1) \leqslant (u'(x))^2.$$

Since

$$e^{u(x)} = 2^{mx-m+1} x^{nx}, \quad u'(x) = n + m \log 2 + n \log x, \quad u''(x) = \frac{n}{x},$$

we have

$$v''(x) \leqslant 0 \;\; \Leftrightarrow \;\; \frac{n}{x}(2^{mx-m+1} x^{nx} - 1) \leqslant (n + m \log 2 + n \log x)^2,$$

therefore $v''(x) \leqslant 0$ is for $x \geqslant 1$ equivalent to

$$2^{mx-m+1}x^{nx} - 1 \leqslant \frac{x}{n}(n + m\log 2 + n\log x)^2.$$

Let $f(x) = 2^{mx-m+1}x^{nx} - 1$ and $g(x) = \frac{x}{n}(n + m\log 2 + n\log x)^2$. Both functions f and g are increasing on $[1, +\infty)$ and $f(1) < g(1)$ because

$$f(1) = 1 \leqslant n = \frac{1}{n} \cdot n^2 < \frac{1}{n}(n + m\log 2)^2 = g(1).$$

By continuity of f we can conclude that there is $M > 1$ such that $f(M) \leqslant g(1)$. For such M

$$f(x) \leqslant f(M) \leqslant g(1) \leqslant g(x), \quad x \in [1, M].$$

This implies that v is concave on $[1, M]$ and consequently

$$v(x) \leqslant v(1) + v'(1)(x-1), \quad x \in [1, M]$$

i.e.

$$\log(2^{mx-m+1}x^{nx} - 1) \leqslant (2m\log 2 + 2n)(x-1), \quad x \in [1, M].$$

The inequality $f(x) \leqslant g(1)$ is equivalent to

$$(mx - m + 1)\log 2 + nx\log x \leqslant \log\left(1 + \frac{(n + m\log 2)^2}{n}\right). \tag{7.26}$$

Because

$$(mx - m + 1)\log 2 + nx\log x \leqslant (mx - m + 1)\log 2 + nx(x-1) \tag{7.27}$$

the inequality (7.26) is the consequence of the inequality

$$(mx - m + 1)\log 2 + nx(x-1) \leqslant \log\left(1 + \frac{(n + m\log 2)^2}{n}\right). \tag{7.28}$$

In (7.27) equality holds only for $x = 1$. Because

$$1 + \frac{(n + m\log 2)^2}{n} > 1 + \frac{n^2}{n} = 1 + n \geqslant 2$$

the inequality (7.28) is a strict inequality for $x = 1$. By this reason, the greater root of the quadratic equation

$$(mx - m + 1)\log 2 + nx(x-1) = \log\left(1 + \frac{(n + m\log 2)^2}{n}\right)$$

is greater than 1. If we denote this root with M the inequality (7.26) holds for $x \in [1, M]$ with equality only for $x = 1$. The first part of Lemma is proved.

Now we prove the second part of the inequality. Both of functions $p(x)$ and $q(x)$ are continuous and increasing. Consequently $r(x) = p^{-1}(x)$ is continuous and increasing. Because

$$p(a_1) = q(a_0) > p(a_0)$$

using monotonicity of $p(x)$ we can conclude that $a_1 > a_0$. Now, by induction and monotonicity of r we can conclude that the sequence a_n is increasing. Now for $x \in [a_n, a_{n+1})$ we have

$$p(x) < p(a_{n+1}) = q(a_n) \leqslant q(x).$$

So $p(x) < q(x)$ holds for $x \in \bigcup_{n=0}^{\infty} [a_n, a_{n+1}) = [a_0, a)$ and using already proved inequality, $p(x) < q(x)$ holds for $1 < x < a$. For $x \geqslant 1$ holds $mx - m + 1 > 1$ and $x^{nx} \geqslant 1$ and consequently

$$p(x) = \log(2^{mx-m+1} x^{nx} - 1) > \log(2\, x^{nx} - 1) \geqslant nx \log x.$$

Because $p(x) > nx \log x \geqslant (n \log x)(x-1)$ inequality $p(c) > q(c)$ holds for c such that $n \log c \geqslant 2m \log 2 + 2n$. It is easy to see that it is true for $c = 2^{\frac{2m}{n}} e^2$. It implies that a is finite (for example $a < 2^{\frac{2m}{n}} e^2$) and a_n is bounded. Letting $n \to \infty$ in $p(a_{n+1}) = q(a_n)$ and using continuity of both functions we conclude that $p(a) = q(a)$. $\square$

7.29. Lemma. *If $a = \varphi_{1/K,n}(1/\sqrt{2})^2$ is as in Theorem 7.3 then for $M > 1$ and $\beta \in [1, M]$*

$$\log\left(\frac{1-a}{a}\right) \le \log(\lambda_n^{2(\beta-1)} 2^\beta - 1) \le V(n)(\beta - 1) \tag{7.30}$$

with $V(n) = (2\log(2\lambda_n^2))(2\lambda_n^2)^{M-1}$ and for $K \in [1, 17]$,

$$\log\left(\frac{1-a}{a}\right) \leqslant (K-1)(4 + 6\log 2) < 9(K-1), \tag{7.31}$$

with equality only for $K = 1$. For $n = 2$

$$\log\left(\frac{1-a}{a}\right) = \log\left(\frac{\varphi_{K,2}(1/\sqrt{2})^2}{\varphi_{1/K,2}(1/\sqrt{2})^2}\right) \leqslant b(K-1) \tag{7.32}$$

where $b = (4/\pi)\,\mathcal{K}(1/\sqrt{2})^2 \le 4.38$.

Proof. For $\beta \in [1, M]$ we have by (7.22)

$$\log\left(\frac{1-a}{a}\right) \le \log(\lambda_n^{2(\beta-1)} 2^\beta - 1)\,.$$

Further, we have

$$\frac{\log(\lambda_n^{2(\beta-1)} 2^\beta - 1)}{\beta - 1} \leqslant 2\,\frac{(2\lambda_n^2)^{\beta-1} - 1}{\beta - 1} \leqslant (2\log(2\lambda_n^2))(2\lambda_n^2)^{M-1}.$$

The second inequality follows from the inequality $\log(t) \leqslant t - 1$ and the third one from Lagrange's theorem and monotonicity of the function $(2\log(2\lambda_n^2))(2\lambda_n^2)^{x-1}$. This proves (7.30).

From (7.23) it follows that the constant a satisfies the inequality

$$a \ge 2^{2(1-K)} K^{-2K} (1/\sqrt{2})^{2K}$$

and also

$$1/a \le 2^{3K-2}K^{2K}\,, \quad K > 1.$$

By Lemma 7.24 we have

$$\log(2^{3K-2}K^{2K} - 1) \leqslant (4 + 6\log 2)(K - 1)$$

for $K \in [1, 17]$ with equality only for $K = 1$. Now, from

$$\frac{1-a}{a} < 2^{3K-2}K^{2K} - 1, \quad K > 1$$

we conclude that

$$\log\left(\frac{1-a}{a}\right) \leqslant (4 + 6\log 2)(K - 1) < 9(K - 1)\,.$$

For the case $n = 2$ we can apply the identity (7.19) and the inequality in (7.20). □

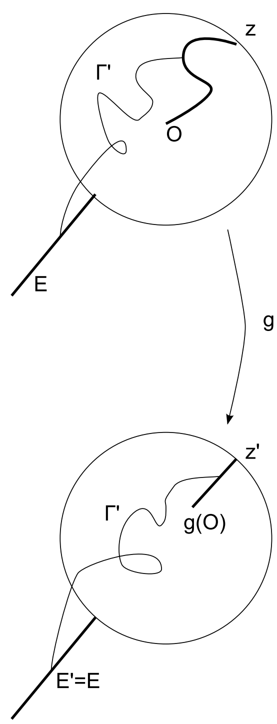

FIGURE 5.

7.33. Proof of Theorem 7.3. Fix $x \in B^n$ and let T_x denote a Möbius transformation of $\overline{\mathbb{R}^n}$ with $T_x(B^n) = B^n$ and $T_x(x) = 0$. Define $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by setting $g(z) = T_x \circ f \circ T_x^{-1}(z)$ for $z \in B^n$ and $g(z) = z$ for $z \in \mathbb{R}^n \setminus B^n$. Then

$g \in Id_K(\partial B^n)$with $g(0) = T_x(f(x))$. By the invariance of ρ_{B^n} under the group $\mathcal{GM}(B^n)$ of Möbius selfautomorphisms of B^n we see that for $x \in B^n$

$$\rho_{B^n}(f(x),x) = \rho_{B^n}(T_x(f(x)),T_x(x)) = \rho_{B^n}(g(0),0). \tag{7.34}$$

Choose $z \in \partial B^n$ such that $g(0) \in [0,z] = \{tz : 0 \leqslant t \leqslant 1\}$. Let $E' = \{-sz : s \geqslant 1\}$, $\Gamma' = \Delta([g(0),z],E';\mathbb{R}^n)$ and $\Gamma = \Delta(g^{-1}[g(0),z],g^{-1}E';\mathbb{R}^n)$.

The spherical symmetrization with center at 0 yields by [**AVV**, Thm 8.44]

$$M(\Gamma) \geqslant \tau_n(1) \quad (= 2^{1-n}\gamma_n(\sqrt{2}))$$

because $g(x) = x$ for $x \in \mathbb{R}^n \setminus B^n$. Next, we see by the choice of Γ' that

$$M(\Gamma') = \tau_n\left(\frac{1+|g(0)|}{1-|g(0)|}\right).$$

By K-quasiconformality we have $M(\Gamma) \leqslant K\,M(\Gamma')$ implying

$$\exp(\rho_{B^n}(0,g(0))) = \frac{1+|g(0)|}{1-|g(0)|} \leqslant \tau_n^{-1}(\tau_n(1)/K) = \frac{1-a}{a}. \tag{7.35}$$

The last equality follows from (7.18). Finally, (7.34) and (7.35) complete the proof. □

7.36. Proof of Theorem 7.4. We have

$$\begin{aligned}
|f(x)-x| &\leqslant 2\tanh\left(\frac{\rho_{B^n}(f(x),x)}{4}\right) \leqslant 2\tanh\left(\frac{\log\left(\frac{1-a}{a}\right)}{4}\right)\\
&\leqslant 2\tanh\left(\frac{(K-1)(4+6\log 2)}{4}\right)\\
&\leqslant (K-1)(2+3\log 2) \leqslant \frac{9}{2}(K-1).
\end{aligned}$$

The first inequality follows from (7.10), the second one from Theorem 7.3, the third one from Lemma 7.29 and the last one from inequality $\tanh(t) \leqslant t$ for $t \geqslant 0$.

For $n = 2$ we use the same first two steps and planar case of Lemma 7.29 to derive inequality

$$|f(x)-x| \leqslant \frac{b}{2}(K-1). \qquad \square$$

A lower bound corresponding to the upper bound in (7.5) is given in the next lemma.

7.37. Lemma. *For $f \in Id(\partial G)$ let*

$$\delta(f) \equiv \sup\{|f(z)-z| : z \in G\}.$$

Then for $f \in Id_K(\partial B^n), K > 1, \alpha = K^{1/(1-n)}$

$$\delta(f) \geq (1-\alpha)\alpha^{\alpha/(1-\alpha)} > \frac{1}{e}(1-\alpha). \tag{7.38}$$

PROOF. The radial stretching $f : B^n \to B^n, n \geq 2$, defined by $f(z) = |z|^{\alpha-1}\, z, z \in B^n$, $(0 < \alpha < 1)$ is K-qc with $\alpha = K^{1/(1-n)}$ [**V2**, p. 49] and $f \in Id_K(\partial B^n)$. Now we have

$$|f(z) - z| = ||z|^{\alpha-1}z - z| = |r^\alpha - r|, \quad |z| = r.$$

Further, we see that

$$\delta(f) = \sup_{0<r<1} (r^\alpha - r),$$

where the supremum is attained for $r = r_\alpha = \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}$, so

$$\delta(f) = (1-\alpha)\alpha^{\alpha/(1-\alpha)}.$$

A crude, but simple, estimate is

$$\delta(f) \geq (1/e)^\alpha - (1/e) = \frac{1}{e}\left(\frac{1}{e^{\alpha-1}} - 1\right) = \frac{1}{e}\left(e^{1-\alpha} - 1\right) \geqslant \frac{1}{e}(1-\alpha).$$

$\square$

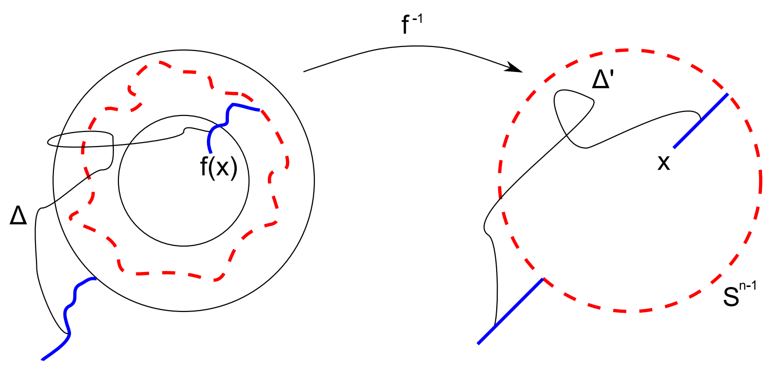

FIGURE 6.

7.39. THEOREM. *Let $f : \overline{\mathbb{R}^n} \longrightarrow \overline{\mathbb{R}^n}$ be a K-qc homeomorphism with $f(\infty) = \infty$ and $B^n(m) \subset f(B^n) \subset B^n(M)$ where $0 < m \leq 1 \leq M$. Then*

$$\eta_{1/K,n}\left(\frac{1+|x|}{1-|x|}\right) \leqslant \frac{M + |f(x)|}{m - |f(x)|}$$

and

$$\frac{m + |f(x)|}{M - |f(x)|} \leqslant \eta_{K,n}\left(\frac{1+|x|}{1-|x|}\right)$$

for all $x \in B^n$ where $\eta_{K,n}(t) = \tau_n^{-1}(\tau_n(t)/K)$.

In particular, if $m = 1 = M$, then we have

$$\eta_{1/K,n}\left(\frac{1+|x|}{1-|x|}\right) \leqslant \frac{1+|f(x)|}{1-|f(x)|} \leqslant \eta_{K,n}\left(\frac{1+|x|}{1-|x|}\right).$$

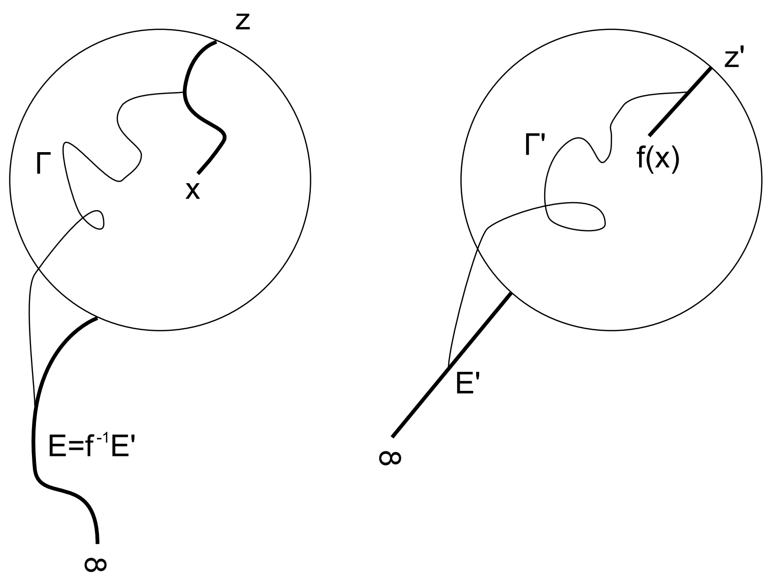

FIGURE 7.

PROOF. The proof is similar to the proof of Theorem 7.3. Fix $x \in B^n$ and choose $z' \in \partial f(B^n)$ such that $f(x) \in [0, z']$ and $[f(x), z') \subset f(B^n)$ and fix $z" \in \partial f(B^n)$ such that $z', 0, z"$ are on the same line, $0 \in [z', z"]$, and $\{-sz" : s \geqslant 1\} \subset \mathbb{R}^n \setminus f(B^n)$. Let $\Gamma' = \Delta([f(x), z'], E'; \mathbb{R}^n)$, $E' = \{-sz" : s \geqslant 1\}$ and $\Gamma = \Delta(f^{-1}[f(x), z'], f^{-1}E'; \mathbb{R}^n)$. Then

$$M(\Gamma') \leq \tau_n \left(\frac{m + |f(x)|}{M - |f(x)|} \right)$$

while applying a spherical symmetrization with center at the origin gives

$$M(\Gamma) \geqslant \tau_n \left(\frac{1 + |x|}{1 - |x|} \right)$$

because $f^{-1}E'$ connects ∂B^n and ∞. Then the inequality $M(\Gamma) \leqslant K\, M(\Gamma')$ yields

$$\tau_n \left(\frac{1 + |x|}{1 - |x|} \right) \leq K \tau_n \left(\frac{m + |f(x)|}{M - |f(x)|} \right),$$

$$\tau_n^{-1} (\frac{1}{K} \tau_n \left(\frac{1 + |x|}{1 - |x|} \right)) \geq \frac{m + |f(x)|}{M - |f(x)|}$$

$$\frac{m + |f(x)|}{M - |f(x)|} \leqslant \eta_{K,n} \left(\frac{1 + |x|}{1 - |x|} \right). \tag{7.40}$$

The lower bound follows if we apply a similar argument to f^{-1} and the lower bound

$$M(\Gamma') \geq \tau_n \left(\frac{M + |f(x)|}{m - |f(x)|} \right).$$

□

7.41. Remark. Putting $x=0, m=1=M$ in (7.40) we obtain by (7.18) for a K-qc homeomorphism $f:\overline{\mathbb{R}^n}\longrightarrow\overline{\mathbb{R}^n}$ with $f(\infty)=\infty$ and $f(B^n)=B^n$ that

$$|f(0)|\le 1-2a\,, a=\varphi_{1/K,n}(1/\sqrt{2})^2\,.$$

Further, if we use the lower bound (7.23) from Lemma 7.21 we obtain

$$|f(0)|\le 1-2^{1-\beta}4^{1-K}K^{-2K}\,.$$

In the special case when $n=2$ we have

$$|f(0)|\le 1-2^{3(1-K)}K^{-2K}\le(2+3\log 2)(K-1)\,.$$

Note that this last inequality does not suppose that $f\in Id_K(\partial B^n)$, only the hypotheses of Theorem 7.39 are needed.

7.42. Maps of cylinder We next consider the class $Id_K(\partial Z)$ for the case when the domain Z is an infinite cylinder.

7.43. THEOREM. *Let* $Z=\{(x,t)\in\mathbb{R}^n\,:\,|x|<1,\,t\in\mathbb{R}\}$, $f\in Id_K(\partial Z)$. *Then* $k_Z(0,f(0))\leqslant c(K)$ *where* $c(K)\to 0$ *when* $K\to 1$.

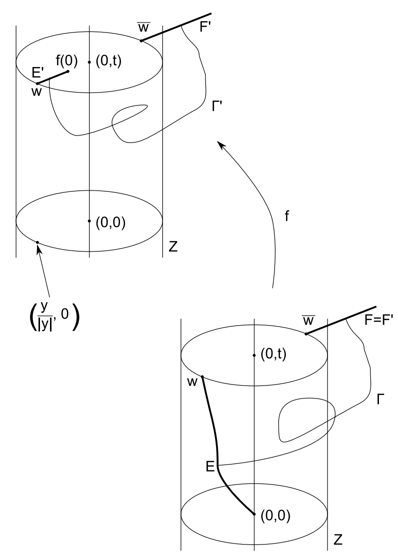

FIGURE 8.

PROOF. Let $f(0)=(y,t)$, $E'=[w,f(0)]$, $F'=\{\overline{w}+s(y,0)\,:\,s\leqslant 0\}$ where $w=(y/|y|,t)$, $\overline{w}=(-y/|y|,t)$. Then E' and F' are the complementary components of a Teichmüller ring and therefore writing $\Gamma'=\Delta(E',F';\mathbb{R}^n)$ we have

$$M(\Gamma')\leqslant\tau_n\left(\frac{1+|y|}{1-|y|}\right).$$

The modulus of the family $\Gamma = \Delta(E, F; \mathbb{R}^n)$, $E = f^{-1}E'$, $F = f^{-1}F'$ can be estimated by use of spherical symmetrization with the center at 0. Note that $E = E'$ because $E' \subset \mathbb{R}^n \setminus Z$ and $f \in Id_K(\partial Z)$. By [**Vu2**, 7.34] we have

$$M(\Gamma) \geqslant \tau_n(1).$$

By K-quasiconformality $M(\Gamma) \leqslant K\, M(\Gamma')$ implying

$$\exp(\rho_{B^{n-1}}(0, y)) = \frac{1 + |y|}{1 - |y|} \leqslant \tau_n^{-1}\left(\frac{\tau_n(1)}{K}\right).$$

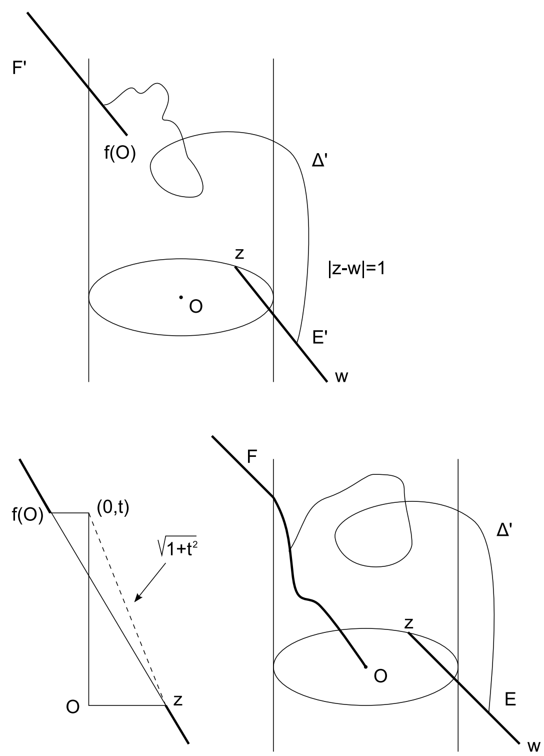

FIGURE 9.

Next we shall estimate t. Fix first z in $\{w \in \partial Z : w_n = 0\}$ such that $|f(0) - z|$ is maximal. Then choose a point w on the line through $f(0)$ and z such that $|z - w| = 1$ and $[z, w] \subset \mathbb{R}^n \setminus Z$. Let $E' = [z, w]$ and $F' = \{f(0) + t(f(0) - z) : t \geqslant 0\}$. Then E' and F' are the complementary components of a Teichmüller ring and with $\Delta' = \Delta(E', F'; \mathbb{R}^n)$ we have

$$M(\Delta') = \tau_n(|f(0) - z|).$$

Observing that $E' = f^{-1}E'$, because $f \in Id_K(\partial Z)$ and carrying out a spherical symmetrization with center at z we see that if $E = f^{-1}E'$, $F = f^{-1}F'$ then

$$M(\Delta) \geqslant \tau_n(1), \quad \Delta = \Delta(E, F; \mathbb{R}^n).$$

By K-quasiconformality we have

$$1+t^2 \leqslant |f(0)-z|^2 \leqslant \tau_n^{-1}\left(\frac{\tau_n(1)}{K}\right)^2.$$

The triangle inequality for k_Z yields

$$\begin{aligned} k_Z(0,f(0)) &\leqslant k_Z(0,(0,t))+k_Z((0,t),(y,t)) \\ &= t+k_{B^{n-1}}(0,y) \leqslant |t|+2\,\rho_{B^{n-1}}(0,y) \\ &\leqslant \sqrt{\tau_n^{-1}\left(\tfrac{\tau_n(1)}{K}\right)^2-1}+2\log\left(\tau_n^{-1}\left(\tfrac{\tau_n(1)}{K}\right)\right) \\ &\leqslant \sqrt{e^{18(K-1)}-1}+18(K-1). \end{aligned}$$

The last inequality follows from (7.18) and Lemma 7.29. □

8. Distortion of two point normalized quasiconformal mappings

Let $\eta\colon [0,\infty)\to[0,\infty)$ be an increasing homeomorphism and $D,D'\subset\mathbb{R}^n$. A homeomorphism $f\colon D\to D'$ is *η-quasisymmetric* if

$$\frac{|f(a)-f(c)|}{|f(b)-f(c)|}\le \eta\left(\frac{|a-c|}{|b-c|}\right) \tag{8.1}$$

for all $a,b,c\in D$ and $c\neq b$. By [**V2**] K-quasiconformal mapping of the whole $\mathbb{R}^n$ is $\eta_{K,n}$- quasisymmetric with a control function $\eta_{K,n}$. Let us define the optimal control function by

$$\eta^*_{K,n}(t)=\sup\{|f(x)|\colon |x|\le t, f\in QC_K(\mathbb{R}^n), f(y)=y \text{ for } y\in\{0,e_1,\infty\}\}.$$

Vuorinen [**Vu2**, Theorem 1.8] proved an upper bound for $\eta^*_{K,n}(t)$, which was later refined by Prause [**P**, Theorem 2.7] for $K<4/3$ into the following form

$$\eta^*_{K,n}(t)\le\begin{cases}\eta^*_{K,n}(1)\varphi_{K,n}(t), & 0<t<1,\\ 1+600\left((K-1)\log\dfrac{1}{K-1}\right), & t=1,\\ \eta^*_{K,n}(1)\dfrac{1}{\varphi_{1/K,n}(1/t)}, & t>1,\end{cases} \tag{8.2}$$

where

$$\eta^*_{K,n}(1)\le \exp((4\sqrt{2}-\log(K-1))(K^2-1)). \tag{8.3}$$

We also introduce a simpler estimate of $\eta^*_{K,n}(1)$ from [**AVV**, Theorem 14.8]

$$\eta^*_{K,n}(1)\le\exp(4K(K+1)\sqrt{K-1}). \tag{8.4}$$

A more rough upper bound for $\eta^*_{K,n}(t)$ can [**Vu1**, Theorem 7.47] be written as

$$\eta^*_{K,n}(t)\le\begin{cases}\eta^*_{K,n}(1)\lambda_n^{1-\alpha}t^{\alpha}, & 0<t\le 1,\\ \eta^*_{K,n}(1)\lambda_n^{1-\beta}t^{\beta}, & t>1,\end{cases} \tag{8.5}$$

where $\alpha = K^{1/(1-n)}$ and $\beta = 1/\alpha$. Furthermore, we can [**Vu1**, Lemma 7.50] estimate

$$\lambda_n^{1-\alpha} \le 2^{1-1/K} K \quad \text{and} \quad \lambda_n^{1-\beta} \le 2^{1-K} K^{-K}. \tag{8.6}$$

8.7. LEMMA. *Let $K \in (1,2]$, $f \in QC_K(\mathbb{R}^n)$, $f(x) = x$ for $x \in \{0, e_1\}$, $\alpha = K^{1/(1-n)}$ and $\beta = 1/\alpha$. Then*

$$\frac{1}{c_3}|x|^\beta \le |f(x)| \le c_3 |x|^\alpha, \quad \text{if } 0 < |x| \le 1,$$
$$\frac{1}{c_3}|x|^\alpha \le |f(x)| \le c_3 |x|^\beta, \quad \text{if } |x| > 1$$

for $c_3 = \exp(60\sqrt{K-1})$.

PROOF. Since f is quasiconformal it is also $\eta^*_{K,n}$-quasisymmetric and by choosing $a = x$, $b = 0$ and $c = e_1$ in (8.1) we have $|f(x)| \le \eta^*_{K,n}(|x|)$. Similarly, selection $(a,b,c) = (e_1, 0, x)$ in (8.1) gives $|f(x)| \ge 1/\eta^*_{K,n}(1/|x|)$. Therefore

$$\frac{1}{\eta^*_{K,n}(1/|x|)} \le |f(x)| \le \eta^*_{K,n}(|x|) \tag{8.8}$$

for all $x \in \overline{\mathbb{R}}^n \setminus \{0\}$. Therefore by (8.5)

$$\frac{1}{c_2}|x|^\beta \le |f(x)| \le c_1 |x|^\alpha, \qquad \text{if } 0 < |x| < 1,$$
$$\frac{1}{\eta^*_{K,n}(1)} \le |f(x)| \le \eta^*_{K,n}(1), \quad \text{if } |x| = 1,$$
$$\frac{1}{c_1}|x|^\alpha \le |f(x)| \le c_2 |x|^\beta, \qquad \text{if } |x| > 1,$$

for $c_1 = \eta^*_{K,n}(1)\lambda_n^{1-\alpha}$ and $c_2 = \eta^*_{K,n}(1)\lambda_n^{1-\beta}$. We can estimate $\max\{c_1, c_2\} \le c_3 = \exp(60\sqrt{K-1})$ for $K \in (1,2]$. $\square$

We will consider K- quasiconformal mapping $f\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ with $f(y) = y$ for $y \in \{0, e_1, \infty\}$ and our goal is to find an upper bound for $|f(x) - x|$ or similar quantities in terms of K and n, when $|x| \le 2$ and $K > 1$ is small enough.

Fix $x \in \mathbb{R}^n \setminus \{0, e_1\}$ and assume that $|x| - \varepsilon \le |f(x)| \le |x| + \varepsilon$ and $|x - e_1| - \varepsilon \le |f(x) - e_1| \le |x - e_1| + \varepsilon$ for $\varepsilon \in (0, \min\{|x|, |x - e_1|\})$. Now

$$|f(x) - x| \le \frac{\operatorname{diam}(A)}{2}, \tag{8.9}$$

where

$$A = A(0, |x| + \varepsilon, |x| - \varepsilon) \cap A(e_1, |x - e_1| + \varepsilon, |x - e_1| - \varepsilon) \cap \{z \in R^3 \colon z_3 = 0\}$$

and

$$A(z, R, r) = B^n(z, R) \setminus \overline{B}^n(z, r).$$

We will now find upper bounds for $\operatorname{diam}(A)$.

8.10. THEOREM. *For $\varepsilon < 1$ and A and x as in (8.9)*

$$\operatorname{diam}(A) \le \sqrt{\varepsilon}\, 4(\min\{|x|, |x - e_1|\} + 1).$$

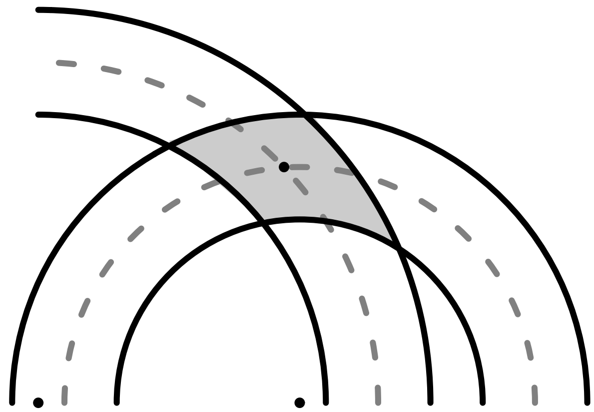

FIGURE 10. The set A.

PROOF. Let us assume $|x| \le |x-e_1|$. Now $\operatorname{diam}(A)$ is maximal when $|x-e_1|/|x|$ is maximal. Therefore we may assume $x = -s$, where $s > 0$. Denote $y = S^1(0, |x| + \varepsilon) \cap S^1(e_1, |x-e_1| - \varepsilon)$. Area of the triangle $\triangle e_1 0 y$ is $(\operatorname{Im} y)/2$ and by Heron's formula

$$\frac{\operatorname{Im} y}{2} = \sqrt{p(p-1)(p-|x|-\varepsilon)(p+\varepsilon-|x|-1)}, \tag{8.11}$$

where $p = |x| + 1$. By (8.11) and the assumption $\varepsilon < 1$ we have

$$\operatorname{Im} y = 2\sqrt{(|x|+1)|x|(1-\varepsilon)\varepsilon} \le 2\sqrt{\varepsilon}(|x|+1)$$

and the assertion follows since $\operatorname{diam}(A) \le 2 \operatorname{Im} y$. □

8.12. THEOREM. *Let A be as in (8.9), $|x| < 2$, $|x - e_1| \le |x|$ and $\measuredangle(1, 0, x) \ge \omega > 0$. Then*

$$\operatorname{diam}(A) \le \varepsilon \left(1 + \frac{70}{\omega}\right)$$

for

$$\varepsilon < \min\left\{1, \frac{1 + |x-e_1| - |x|}{2}, \frac{|x| + |x-e_1| - 1}{2}\right\}.$$

PROOF. Let us denote by y the intersection of $S^1(|x| + \varepsilon)$ and $S^1(e_1, |x - e_1|)$ in the first quadrant. The triangles $\triangle(0, 1, x)$ and $\triangle(0, 1, y)$ give by the Law of Cosines

$$|x-1|^2 = |x|^2 + 1 - 2|x| \cos\gamma$$

and

$$|y-1|^2 = |y|^2 + 1 - 2|y| \cos\delta,$$

where η is the angle $\measuredangle(1, 0, x)$ and ξ is the angle $\measuredangle(1, 0, y)$. Therefore

$$\cos\gamma = \frac{|x|^2 + 1 - |x-1|^2}{2|x|} \tag{8.13}$$

and

$$\cos\delta = \frac{|y|^2+1-|y-1|^2}{2|y|} = \frac{(|x|+\varepsilon)^2+1-|x-1|^2}{2(|x|+\varepsilon)}. \tag{8.14}$$

By the Jordan inequality

$$|\cos\gamma-\cos\delta| = \cos\delta-\cos\gamma = 2\sin\frac{\delta+\gamma}{2}\sin\frac{\gamma-\delta}{2} \geq \frac{2}{\pi^2}(\gamma+\delta)(\gamma-\delta)$$

and by assumption

$$|\gamma-\delta| \leq \frac{\pi^2}{2\omega}|\cos\gamma-\cos\delta|. \tag{8.15}$$

By the triangle inequality, the Jordan inequality, (8.15), (8.13) and (8.14)

$$\begin{aligned}
|x-y| &\leq \varepsilon + 2|x|\sin\frac{|\gamma-\delta|}{2} \\
&\leq \varepsilon + \frac{2|x|}{\pi}|\gamma-\delta| \\
&\leq \varepsilon + \frac{|x|\pi}{\omega}|\cos\gamma-\cos\delta| \\
&= \varepsilon + \frac{|x|\pi}{\omega}\frac{\varepsilon(1+|x-1|^2+|x|(|x|+\varepsilon))}{2|x|(|x|+\varepsilon)} \\
&\leq \varepsilon + \frac{\pi}{\omega}\frac{\varepsilon(1+3^2+2(2+1))}{2(1/2+0)} \\
&= \varepsilon + \frac{51\varepsilon}{\omega}.
\end{aligned}$$

Let us denote by z the intersection of $S^1(|x|+\varepsilon)$ and $S^1(e_1,|x-e_1|+\varepsilon)$ in the first quadrant. If $\delta > \omega/2$, then we obtain

$$|x-y| \leq \frac{|x|\pi}{\omega}\frac{\varepsilon||y-1|^2-|z-1|^2|}{2(|x|+\varepsilon)} \leq \frac{19\varepsilon}{\omega}.$$

Now

$$\operatorname{diam}(A) \leq |x-z| \leq |x-y|+|y-z| \leq \varepsilon + \frac{70\varepsilon}{\omega}$$

and the assertion follows. $\square$

8.16. LEMMA. *Let $n \geq 2$, $K > 1$, $\alpha = K^{1/(1-n)}$, $\beta = 1/\alpha$ and $c_3 = \exp(60\sqrt{K-1})$. For $t \in (0,1)$*

$$c_3 t^\alpha - t \geq t - \frac{t^\beta}{c_3} \tag{8.17}$$

and for $t > 1$

$$c_3 t^\beta - t \geq t - \frac{t^\alpha}{c_3}. \tag{8.18}$$

PROOF. To prove (8.17) it is sufficient to prove that $f(t) \geq 0$, where $f(t) = c_3 t^{\alpha} + \frac{1}{c_3} t^{1/\alpha} - 2t$ and $0 < t < 1$. Because

$$\lim_{t \to 0+} f(t) = 0,$$

it is sufficient to prove $f'(t) \geq 0$ for $0 < t < 1$, i.e.

$$\alpha c_3 t^{\alpha-1} + \frac{1}{\alpha c_3} t^{(1/\alpha)-1} - 2 \geq 0. \tag{8.19}$$

Using inequality between arithmetic and geometric means, we can conclude that

$$\alpha c_3 + \frac{1}{\alpha c_3} \geq 2$$

holds. In other words,

$$\lim_{t \to 1-} f'(t) = \alpha c_3 + \frac{1}{\alpha c_3} - 2 \geq 0.$$

By this reason, to prove inequality (8.19) it is sufficient to prove that $f''(t) \leq 0$ for $0 < t < 1$ i.e.

$$\alpha(\alpha - 1) c_3 t^{\alpha-2} + \frac{\frac{1}{\alpha} - 1}{\alpha c_3} t^{(1/\alpha)-2} \leq 0,$$

or equivalently

$$t^{\frac{1}{\alpha} - \alpha} \leq \alpha^3 c_3^2.$$

The last inequality follows from

$$t^{\frac{1}{\alpha} - \alpha} < 1 \leq \alpha^3 c_3^2.$$

The first inequality holds because $0 < t < 1$ and $\frac{1}{\alpha} - \alpha > 0$ (because $0 < \alpha < 1$). Now we prove $\alpha^3 c_3^2 \geq 1$ to complete proof. This is same inequality as

$$K^{3/(1-n)} e^{120\sqrt{K-1}} \geq 1,$$

or equivalently

$$e^{40(n-1)u} \geq u^2 + 1$$

for $u = \sqrt{K-1}$. Because $u \geq 0$, using Taylor series for e^x we can conclude that

$$e^{40(n-1)u} \geq 1 + 40(n-1)u + \frac{(40(n-1))^2 u^2}{2} \geq 1 + u^2.$$

The inequality (8.18) is equivalent to

$$c_3 t^{(1/\alpha)-1} + \frac{t^{\alpha-1}}{c_3} \geq 2. \tag{8.20}$$

Inequality (8.20) holds for $t = 1$. To prove inequality (8.20) for $t > 1$ it is sufficient to prove that derivation of the left side of inequality is nonnegative. We have following sequence of equivalent formulas:

$$c_3 \left(\frac{1}{\alpha} - 1 \right) t^{\frac{1}{\alpha} - 2} + \frac{\alpha - 1}{c_3} t^{\alpha - 2} \geq 0$$

$$\frac{(1-\alpha)c_3}{\alpha}\, t^{\frac{1}{\alpha}-2} \geq \frac{1-\alpha}{c_3}\, t^{\alpha-2}$$

$$t^{\frac{1}{\alpha}-\alpha} \geq \frac{\alpha}{c_3^2}.$$

The last inequality is true because

$$t^{\frac{1}{\alpha}-\alpha} \geq 1 \geq \frac{\alpha}{c_3^2}$$

and the assertion follows. □

8.21. LEMMA. *Let* $\varepsilon > 0$. *Then*

$$|x| - \varepsilon \leq |f(x)| \leq |x| + \varepsilon$$

for

$$1 < K \leq \max\left\{ \left(\frac{\log(\varepsilon+1)}{60}\right)^2 + 1, 2 \right\}.$$

PROOF. Let us denote $l(x) = c_3^{-1} \max\{|x|^\alpha, |x|^\beta\}$ and $u(x) = c_3 \max\{|x|^\alpha, |x|^\beta\}$. We will first consider the case $0 < |x| < 1$. By Lemma 8.16

$$\begin{aligned} \max\{u(x) - |x|, |x| - l(x)\} &= \max\left\{ c_3|x|^\alpha - |x|, |x| - \frac{1}{c_3}|x|^\beta \right\} \\ &= c_3|x|^\alpha - |x| \\ &\leq \exp(60\sqrt{K-1})|x|^\alpha - |x| \\ &\leq \exp(60\sqrt{K-1})|x|^{1/K} - |x| \\ &\leq \exp(60\sqrt{K-1}) - 1. \end{aligned}$$

Now $\exp(60\sqrt{K-1}) - 1 \leq \varepsilon$ is equivalent to

$$K \leq \left(\frac{\log(\varepsilon+1)}{60}\right)^2 + 1. \tag{8.22}$$

If $|x| = 1$, then

$$\max\{u(x) - |x|, |x| - l(x)\} = c_3 - 1$$

and therefore we want $\exp(60\sqrt{K-1}) - 1 \leq \varepsilon$ for $K \in (1, 2]$, which is equivalent to

$$K \leq \left(\frac{\log(\varepsilon+1)}{60}\right)^2 + 1. \tag{8.23}$$

Let us first consider the case $1 < |x| < 2$. By Lemma 8.16

$$\begin{aligned}\max\{u(x)-|x|,|x|-l(x)\} &= \max\left\{c_3|x|^\beta-|x|,|x|-\frac{1}{c_3}|x|^\alpha\right\}\\ &= c_3|x|^\beta-|x|\\ &\le \exp(60\sqrt{K-1})|x|^\beta-|x|\\ &\le \exp(60\sqrt{K-1})|x|^K-|x|\\ &\le |x|(\exp(60\sqrt{K-1})|x|^{K-1}-1)\\ &\le 2(\exp(60\sqrt{K-1}+(K-1)\log|x|)-1)\\ &\le 2(\exp(60(K-1)^{3/2}-1).\end{aligned}$$

Now $2(\exp(60(K-1)^{3/2}-1)\le\varepsilon$ is equivalent to

$$K \le \left(\frac{\log(\varepsilon/2+1)}{60}\right)^{2/3}+1. \tag{8.24}$$

By combining (8.22), (8.23) and (8.24) we have

$$|x|-\varepsilon \le |f(x)| \le |x|+\varepsilon$$

for

$$K \le \min\left\{\left(\frac{\log(\varepsilon+1)}{60}\right)^2+1,2,\left(\frac{\log(\varepsilon/2+1)}{60}\right)^{2/3}+1\right\} = \left(\frac{\log(\varepsilon+1)}{60}\right)^2+1$$

and the assertion follows. □

8.25. LEMMA. *For* $0<\alpha<1$, $c>1$ *and* $t>0$

$$\log(1+c\max\{t^\alpha,t^{1/\alpha}\}) \le \begin{cases}\frac{c}{\alpha}\log^\alpha(1+t), & 0<t<1,\\ \frac{c}{\alpha}\log(1+t), & t\ge 1.\end{cases}$$

PROOF. Let us first assume $t\ge 1$. Then $\max\{t^\alpha,t^{1/\alpha}\}=t^{1/\alpha}$ and by the generalized Bernoulli inequality

$$(1+t)^{c/\alpha} \ge (1+ct)^{1/\alpha} \ge 1+c^{1/\alpha}t^{1/\alpha} \ge 1+ct^{1/\alpha}$$

implying $\log(1+ct^{1/\alpha}) \le c/\alpha\log(1+t)$.

Let us then assume $0<t<1$. Now $\max\{t^\alpha,t^{1/\alpha}\}=t^\alpha$, we will show that function

$$f(t)=\log(1+ct^\alpha)-\frac{c}{\alpha}\log^\alpha(1+t)$$

is nonpositive. We easily obtain

$$f'(t)=\frac{\alpha ct^{\alpha-1}}{1+ct^\alpha}-\frac{c\log^{\alpha-1}(1+t)}{1+t}. \tag{8.26}$$

Since $\alpha-1<0$ and $\log(1+t)\le t$ we have

$$t^{\alpha-1} \le \log^{\alpha-1}(1+t). \tag{8.27}$$

By assumptions $c > 1 \geq t^{1-\alpha}$ and therefore

$$ct^{\alpha} \geq t. \tag{8.28}$$

By (8.26), (8.27) and (8.28) $f'(t) \leq 0$ is equivalent to $(1-\alpha)(1+t) \geq 0$ and therefore $f(t)$ is increasing. Now we have $f(t) \leq f(0) = 0$ and the assertion follows. □

8.29. THEOREM. *Let $G = \mathbb{R}^n \setminus \{0\}$, $f \in QC_K$ and $f(0) = 0$. There exists $c(K)$ such that*

$$j_G(f(x), f(y)) \leq c(K) \max\{j_G(x,y)^{\alpha}, j_G(x,y)\},$$

where $\alpha = K^{1/(1-n)}$, and $c(K) \to 1$ as $K \to 1$.

PROOF. By symmetry we may assume $x = e_1$ and $|y| \geq 1$. Now

$$\frac{|f(y) - f(e_1)|}{|e_1|} = \frac{|f(y) - f(x)|}{|f(0) - f(e_1)|} \leq \eta\left(\frac{|x-y|}{|0-e_1|}\right) = \eta(|x-y|)$$

and

$$\frac{|f(y) - f(e_1)|}{|f(y)|} = \frac{|f(y) - f(x)|}{|f(y) - f(0)|} \leq \eta\left(\frac{|x-y|}{|y-0|}\right) = \eta\left(\frac{|x-y|}{|y|}\right).$$

Therefore by Lemma 8.25

$$\begin{aligned}
j(f(x), f(y)) &= \log\left(1 + \frac{|f(x) - f(y)|}{\min\{|f(x)|, |f(y)|\}}\right) \\
&= \log\left(1 + \max\left\{\eta(|y-e_1|), \eta\left(\frac{|x-y|}{|y|}\right)\right\}\right) \\
&= \log(1 + \eta(|y-e_1|)) \\
&\leq \log(1 + c_3 \max\{|y-e_1|^{\alpha}, |y-e_1|^{1/\alpha}\}) \\
&\leq \begin{cases} \frac{c_3}{\alpha} \log^{\alpha}(1 + |y-e_1|), & 0 < |y-e_1| < 1, \\ \frac{c_3}{\alpha} \log(1 + |y-e_1|), & |y-e_1| \geq 1. \end{cases}
\end{aligned}$$

By choosing $c(K) = c_3/\alpha$ we have $c(K) \to 1$ as $K \to 1$ by Lemma 8.7 and the assertion follows. □

8.30. PROBLEM. Is this Theorem true for k_G instead of j_G?

CHAPTER 2

Harmonic Quasiregular Mappings

It is well known that if f is a complex-valued harmonic function defined in a region G of the complex plane $\mathbb{C}$, then $|f|^p$ is subharmonic for $p \geq 1$, and that in the general case is not subharmonic for $p < 1$. However, if f is holomorphic, then $|f|^p$ is subharmonic for every $p > 0$. Here we consider k-quasiregular harmonic functions $(0 < k < 1)$. We recall that a harmonic function is quasiregular if

$$|\bar{\partial} f(z)| \leq k|\partial f(z)|, \qquad z \in G,$$

where

$$\bar{\partial} f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \partial f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad z = x + iy.$$

We prove that $|f|^p$ is subharmonic for $p \geq 4k/(1+k)^2 =: q$ as well as that the exponent q (< 1) is the best possible (see Theorem 1.1). The fact that $q < 1$ enables us to prove that if f is quasiregular in the unit disk $\mathbb{D}$ and continuous on $\overline{D}$, then $\tilde{\omega}(f, \delta) \leq \text{const.}\omega(f, \delta)$, where $\tilde{\omega}(f, \delta)$ (respectively $\omega(f, \delta)$) denotes the modulus of continuity of f on $\mathbb{D}$ (respectively $\partial\mathbb{D}$); see Theorem 2.1.

1. Subharmonicity of $|f|^p$

1.1. Theorem. **[KP]** *If f is a complex-valued k-quasiregular harmonic function defined on a region $G \subset \mathbb{C}$, and $q = 4k/(k+1)^2$, then $|f|^q$ is subharmonic. The exponent q is optimal.*

Recall that a continuous function u defined on a region $G \subset \mathbb{C}$ is subharmonic if for all $z_0 \in G$ there exists $\varepsilon > 0$ such that

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\, dt, \qquad 0 < r < \varepsilon, \tag{1.2}$$

If $u(z_0) = |f(z_0)|^2 = 0$, then (1.2) holds. If $u(z_0) > 0$, then there exists a neighborhood U of z_0 such that u is of class $C^2(U)$ (because the zeroes of u are isolated), and then we may prove that $\Delta u \geq 0$ on U. Thus the proof reduces to proving that $\Delta u(z) \geq 0$ whenever $u(z) > 0$. In order to do this we will calculate Δu.

It is easy to prove that If $u > 0$ is a C^2 function defined on a region in $\mathbb{C}$, and $\alpha \in \mathbb{R}$, then next two statements holds

$$\Delta(u^\alpha) = \alpha u^{\alpha-1}\Delta u + \alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2, \tag{1.3}$$

$$|\nabla u|^2 = 4|\partial u|^2 \quad \text{and} \quad \Delta u = 4\partial\bar{\partial} u. \tag{1.4}$$

1.5. Lemma. *If $f = g + \bar{h}$, where g and h are holomorphic functions, then*

$$\Delta(|f|^2) = 4(|g'|^2 + |h'|^2). \tag{1.6}$$

Proof. Since $|f|^2 = (g + \bar{h})(\bar{g} + h)$, we have

$$\begin{aligned}\Delta(|f|^2) &= 4\partial(\overline{h'}(\bar{g} + h) + (g + \bar{h})\overline{g'})\\ &= 4(\overline{h'}h + g\overline{g'})\\ &= 4(|g'|^2 + |h'|^2).\end{aligned}$$

□

1.7. Lemma. *If $f = g + \bar{h}$, where g and h are holomorphic functions, then*

$$|\nabla(|f|^2)|^2 = 4(|\overline{g'}|^2 + |\overline{h'}|^2)|f|^2 + 8\operatorname{Re}(\overline{g'}h'f^2). \tag{1.8}$$

Proof. We have

$$\begin{aligned}|\nabla(|f|^2)|^2 &= 4|\partial(|f|^2)|^2\\ &= 4|\partial((g + \bar{h})(\bar{g} + h))|^2\\ &= 4|g'\bar{f} + fh'|^2\\ &= 4(|g'|^2 + |h'|^2)|f|^2 + 8\operatorname{Re}(\overline{g'}h'f^2).\end{aligned}$$

□

1.9. Lemma. *If $f = g + \bar{h}$, where g and h are holomorphic functions, then*

$$\Delta(|f|^p) = p^2(|g'|^2 + |h'|^2)|f|^{p-2} + 2p(p-2)|f|^{p-4}\operatorname{Re}(\overline{g'}h'f^2) \tag{1.10}$$

whenever $f \neq 0$.

Proof. We take $\alpha = p/2$, $u = |f|^2$, and then use (1.3), (1.6) and (1.8) to get the result. □

1.11. Proof of Theorem 1.1. We have to prove that $\Delta(|f|^p) \geq 0$, where $p = 4k/(1+k)^2$. Since $p - 2 < 0$, we get from (1.10) that

$$\begin{aligned}\Delta(|f|^p) &\geq p^2(|g'|^2 + |h'|^2)|f|^{p-2} + 2p(p-2)|f|^{p-4}|g'| \cdot |h'| \cdot |f|^2\\ &= p^2|g'|^2(m^2+1)|f|^{p-2} + 2p(p-2)|g'|^2|f|^{p-2}m\\ &= p|g'|^2|f|^{p-2}[p(1+m^2) + 2(p-2)m],\end{aligned}$$

where $m = |h'|/|g'| \leq k$. The function $m \mapsto p(1+m^2) + 2(p-2)m$ has a negative derivative (because $p < 1$ and $m < 1$), which implies that

$$(1+m^2)p + 2(p-2)m \geq (1+k^2)p + 2(p-2)k.$$

On the other hand $(1+k^2)p + 2(p-2)k \geq 0$ if and only if $p \geq 4k/(1+k)^2$, which proves that $|f|^q$ is subharmonic. To prove that the exponent q is optimal we take $f(z) = z + k\bar{z}$. By (1.10),

$$\Delta(|f|^p)(1) = p^2(1+k^2)(1+k)^{p-2} + 2p(p-2)(1+k)^{p-2}k.$$

Hence $\Delta(|f|^p)(1) \geq 0$ if and only if

$$p(1+k^2) + 2(p-2)k \geq 0,$$

which, as noted above, is equivalent to $p \geq q$. This completes the proof of Theorem 1.1. $\square$

2. Moduli of continuity in Euclidean metric

For a continuous function $f : \overline{\mathbb{D}} \mapsto \mathbb{C}$ harmonic in $\mathbb{D}$ we define two moduli of continuity:

$$\omega(f,\delta) = \sup\{|f(e^{i\theta}) - f(e^{it})| : |e^{i\theta} - e^{it}| \leq \delta,\ t,\theta \in \mathbb{R}\}, \quad \delta \geq 0,$$

and

$$\tilde{\omega}(f,\delta) = \sup\{|f(z) - f(w)| : |z-w| \leq \delta,\ z,w \in \overline{\mathbb{D}}\}, \quad \delta \geq 0.$$

Clearly $\omega(f,\delta) \leq \tilde{\omega}(f,\delta)$, but the reverse inequality need not hold. To see this consider the function

$$f(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{(-1)^n r^n \cos n\theta}{n^2}, \qquad re^{i\theta} \in \overline{\mathbb{D}}.$$

This function is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. The function $v(\theta) = f(e^{i\theta})$, $|\theta| < \pi$, is differentiable, and

$$\begin{aligned} \frac{dv}{d\theta} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin n\theta}{n} \\ &= \frac{\theta}{2}, \qquad |\theta| < \pi. \end{aligned}$$

This formula is well known, and can be verified by calculating the Fourier coefficients of the function $\theta \mapsto \theta/2$, $|\theta| < \pi$. It follows that

$$|f(e^{i\theta}) - f(e^{it})| \leq (\pi/2)|\theta - t|, \qquad -\pi < \theta,\ t < \pi,$$

and hence $\omega(f,\delta) \leq M\delta$, $\delta > 0$, where M is an absolute constant. On the other hand, the inequality $\tilde{\omega}(f,\delta) \leq CM\delta$, $C = \text{const}$, does not hold because it implies that $|\partial f/\partial r| \leq CM$, which is not true because

$$\frac{\partial}{\partial r} f(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{r^{n-1}}{n}, \quad \text{for } \theta = \pi,\ 0 < r < 1.$$

However, as was proved by Rubel, Shields and Taylor [**RST**], and Tamrazov [**TA**], if f is a holomorphic function, then $\tilde{\omega}(f,\delta) \leq C\omega(f,\delta)$, where C is independent of f and δ. Here we extend that result to quasiregular harmonic functions.

2.1. THEOREM. [**KP**] *Let f be a k-quasiregular harmonic complex-valued function which has a continuous extension on $\overline{\mathbb{D}}$, then there is a constant C depending only on k such that $\tilde{\omega}(f,\delta) \leq C\omega(f,\delta)$.*

In order to deduce this fact from Theorem 1.1, we need some simple properties of the modulus $\omega(f,\delta)$. Let

$$\omega_0(f,\delta) = \sup\{|f(e^{i\theta}) - f(e^{it})| : |\theta - t| \le \delta,\ t, \theta \in \mathbb{R}\}.$$

It is easy to check that

$$C^{-1}\omega_0(f,\delta) \le \omega(f,\delta) \le C\omega_0(f,\delta), \tag{2.2}$$

where C is an absolute constant, and that

$$\omega_0(f,\delta_1+\delta_2) \le \omega_0(f,\delta_1) + \omega_0(f,\delta_2), \quad \delta_1,\ \delta_2 \ge 0.$$

Hence, $\omega_0(f,2^n\delta) \le 2^n\omega_0(f,\delta)$, and hence $\omega_0(\lambda\delta) \le 2\lambda\omega_0(\delta)$, for $\lambda \ge 1, \delta \ge 0$. From these inequalities and (2.2) it follows that

$$\omega(f,\lambda\delta) \le 2C\lambda\omega(f,\delta), \quad \lambda \ge 1, \delta \ge 0, \tag{2.3}$$

and

$$\omega(f,\delta_1+\delta_2) \le C\omega(f,\delta_1) + C\omega(f,\delta_2), \quad \delta_1,\ \delta_2 \ge 0, \tag{2.4}$$

where C is an absolute constant. As a consequence of (2.3) we have, for $0 < p < 1$,

$$\int_x^\infty \frac{\omega(f,t)^p}{t^2}\,dt \le C\frac{\omega(f,x)^p}{x}, \quad x > 0, \tag{2.5}$$

where C depends only on p. Finally we need the following consequence of the harmonic Schwarz lemma (see **[ABR]**).

2.6. Lemma. *If h is a function harmonic and bounded in the unit disk, with $h(0) = 0$, the $|h(\xi)| \le (4/\pi)\|h\|_\infty|\xi|$, for $\xi \in \mathbb{D}$.*

2.7. Proof of Theorem 2.1. It is enough to prove that $|f(z)-f(w)| \le C\omega(f,|z-w|)$ for all z, $w \in \overline{\mathbb{D}}$, where C depends only on k. Assume first that $z = r \in (0,1)$ and $|w| = 1$. Then, by Theorem 1.1, the function $\varphi(\xi) = |f(w) - f(\xi)|^q$, where $q = 4k/(1+k)^2 < 1$, is subharmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, whence

$$\varphi(r) \le \frac{1}{2\pi}\int_{\partial\mathbb{D}} \frac{(1-r^2)\varphi(\zeta)}{|\zeta - r|^2}\,|d\zeta|.$$

Since, by (2.4),

$$\begin{aligned}\varphi(\zeta) &\le (\omega(f,|w-r| + |r-\zeta|))^q \\ &\le C^q\omega(f,|w-r|)^q + C^q\omega(f,|r-\zeta|)^q,\end{aligned}$$

we have

$$\begin{aligned}\varphi(z) &\le C^q\omega(f,|w-r|)^q + \frac{C^q}{2\pi}\int_{\partial\mathbb{D}} \frac{(1-r^2)\omega(f,|r-\zeta|)^q}{|\zeta-r|^2}\,|d\zeta| \\ &= C^q\omega(f,|w-r|)^q + \frac{C^q}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,\omega(|r-e^{it}|)^q}{|e^{it}-r|^2}\,dt.\end{aligned}$$

But simple calculation shows that

$$|r-e^{it}| = \sqrt{(1-r)^2+4r\sin^2(t/2)} \asymp 1-r+|t| \quad (0<r<1,\ |t|\le\pi).$$

From this, (1.2), and (2.5) it follows that

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{(1-r^2)\,\omega(f,|r-e^{it}|)^q}{|e^{it}-r|^2}\,dt &\le C_1\int_0^{\pi}\frac{(1-r)\,\omega(f,1-r+t)^q}{(1-r+t)^2}\,dt\\ &= C_1\Big(\int_0^{1-r}+\int_{1-r}^{\pi}\Big)\frac{(1-r)\,\omega(f,1-r+t)^q}{(1-r+t)^2}\,dt\\ &\le C_2\,(\omega(1-r))^q + C_2\,(1-r)\int_{1-r}^{\infty}\frac{\omega(f,t)^q}{t^2}\,dt\\ &\le C_3\,(\omega(f,1-r))^q\\ &\le C_4\,(\omega(f,|w-z|))^q.\end{aligned}$$

Thus $|f(w)-f(z)|\le C_5\omega(f,|w-z|)$ provided $w\in\partial\mathbb{D}$ and $z\in(0,1)$. By rotation and the continuity of f, we can extend this inequality to the case where $w\in\partial\mathbb{D}$ and $z\in\overline{\mathbb{D}}$.

If $0<|w|<1$, we consider the function $h(\xi)=f(\xi w/|w|)-f(\xi z/|w|)$, $|\xi|\le 1$. This function is harmonic in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and $h(0)=0$. Hence, by the harmonic Schwarz lemma, inequality (1.2), and the preceding case,

$$\begin{aligned}|f(w)-f(z)| &= |h(|w|)|\\ &\le (4/\pi)|w|\,\|h\|_\infty\\ &\le C_6|w|\,\omega(f,|w/|w|-z/|w|\,|)\\ &\le C_7\,\omega(f,|w|\,|\,w/|w|-z/|w|\,|)\\ &= C_7\,\omega(f,|w-z|),\end{aligned}$$

which completes the proof. □

3. Lipschitz continuity up to the boundary on B^n

It is known, even for $n=2$, that Lipschitz continuity of $\phi: T\to C$, where $T=\{z\in C: |z|=1\}$, does not imply Lipschitz continuity of $u=P[\phi]$.

Here, for any $n\ge 2$,

$$P[\phi](x)=\int_{S^{n-1}}P(x,\xi)\phi(\xi)d\sigma(\xi),\ \ x\in B^n$$

where $P(x,\xi)=\frac{1-|x|^2}{|x-\xi|^n}$ is the Poisson kernel for the unit ball $B^n=\{x\in R^n: |x|<1\}$, $d\sigma$ is the normalized surface measure on the unit sphere S^{n-1} and $\phi: S^{n-1}\to R^n$ is a continuous mapping.

Our aim is to show that Lipschitz continuity is preserved by harmonic extension, if the extension is quasiregular. The analogous statement is true for Hölder continuity without assumption of quasiregularity.

3.1. THEOREM. [**AKM**] *Assume* $\phi : S^{n-1} \to R^n$ *satisfies a Lipschitz condition:*

$$|\phi(\xi) - \phi(\eta)| \le L|\xi - \eta|, \quad \xi, \eta \in S^{n-1}$$

and assume $u = P[\phi] : B^n \to R^n$ *is* K*-quasiregular. Then*

$$|u(x) - u(y)| \le C'|x - y|, \quad x, y \in B^n$$

where C' *depends on* L*,* K *and* n *only.*

D. Kalaj obtained a related result, but under additonal assumption of $C^{1,\alpha}$ regularity of ϕ, (see [**KA**]).

PROOF. The main part of the proof is the estimate of the tangential derivatives of u, and in that part quasiregularity plays no role. We choose $x_0 = r\xi_0 \in B^n$, $r = |x|$, $\xi_0 \in S^{n-1}$. Let $T = T_{x_0} rS^{n-1}$ be the $n-1$ dimensional tangent plane at x_0 to the sphere rS^{n-1}. We want to prove that

$$\|D(u|_T)(x_0)\| \le C(n)L. \tag{3.2}$$

Without loss of generality we can assume $\xi_0 = e_n$ and $x_0 = re_n$. By a simple calculation

$$\frac{\partial}{\partial x_j} P(x, \xi) = \frac{-2x_j}{|x - \xi|^n} - n(1 - |x|^2)\frac{x_j - \xi_j}{|x - \xi|^{n+2}}.$$

Hence, for $1 \le j < n$ we have

$$\frac{\partial}{\partial x_j} P(x_0, \xi) = n(1 - |x_0|^2)\frac{\xi_j}{|x_0 - \xi|^{n+2}}.$$

It is important to note that this kernel is odd in ξ (with respect to reflection $(\xi_1, \dots \xi_j, \dots, \xi_n) \mapsto (\xi_1, \dots, -\xi_j, \dots, \xi_n)$), a typical fact for kernels obtained by differentiation. This observation and differentiation under integral sign gives, for any $1 \le j < n$,

$$\begin{aligned} \frac{\partial u}{\partial x_j}(x_0) &= n(1 - r^2) \int_{S^{n-1}} \frac{\xi_j}{|x_0 - \xi|^{n+2}} \phi(\xi) d\sigma(\xi) \\ &= n(1 - r^2) \int_{S^{n-1}} \frac{\xi_j}{|x_0 - \xi|^{n+2}} (\phi(\xi) - \phi(\xi_0)) d\sigma(\xi). \end{aligned}$$

Using the elementary inequality $|\xi_j| \le |\xi - \xi_0|$, $(1 \le j < n,\ \xi \in S^{n-1})$ and Lipschitz continuity of ϕ we get

$$\begin{aligned} \left| \frac{\partial u}{\partial x_j}(x_0) \right| &\le Ln(1 - r^2) \int_{S^{n-1}} \frac{|\xi_j||\xi - \xi_0|}{|x_0 - \xi|^{n+2}} d\sigma(\xi) \\ &\le Ln(1 - r^2) \int_{S^{n-1}} \frac{|\xi - \xi_0|^2}{|x_0 - \xi|^{n+2}} d\sigma(\xi). \end{aligned}$$

In order to estimate the last integral, we split S^{n-1} into two subsets $E = \{\xi \in S^{n-1} : |\xi - \xi_0| \le 1 - r\}$ and $F = \{\xi \in S^{n-1} : |\xi - \xi_0| > 1 - r\}$. Since $|\xi - x_0| \ge 1 - |x_0|$ for all $\xi \in S^{n-1}$ we have

$$\begin{aligned}
\int_E \frac{|\xi - \xi_0|^2}{|x_0 - \xi|^{n+2}} d\sigma(\xi) &\le (1 - r^2)^{-n-2} \int_E |\xi - \xi_0|^2 d\sigma(\xi) \\
&\le (1 - r^2)^{-n-2} \int_0^{1-r} \rho^2 \rho^{n-2} d\rho \\
&\le \frac{2}{n+1} (1 - r)^{-1}.
\end{aligned}$$

On the other hand, $|\xi - \xi_0| \le C_n |\xi - x_0|$ for every $\xi \in F$, so

$$\begin{aligned}
\int_F \frac{|\xi - \xi_0|^2}{|x_0 - \xi|^{n+2}} d\sigma(\xi) &\le C_n^{n+2} \int_F |\xi - \xi_0|^{-n} d\sigma(\xi) \\
&\le C_n' \int_{1-r}^{2} \rho^{-n} \rho^{n-2} d\rho \\
&\le C_n' (1 - r)^{-1}.
\end{aligned}$$

Combining these two estimates we get

$$\left| \frac{\partial u}{\partial x_j}(x_0) \right| \le LC(n)$$

for $1 \le j < n$. Due to rotational symmetry, the same estimate holds for every derivative in any tangential direction. This establishes estimate (3.2). Finally, K-quasiregularity gives

$$\|Du(x)\| \le LKC(n).$$

Now the mean value theorem gives Lipschitz continuity of u. $\square$

3.3. Problem. (1) Can one prove similar result for other type of moduli of continuity, as was done in Section 2 in the planar case?
(2) The same questions can be posed in other smoothly bounded domains.

4. Bilipschitz maps

Bilipschitz property of harmonic quasiconformal mappings on the unit disc was investigated in **[MAT]**. A different approach to the following theorem is given in **[MAT1]**.

4.1. Theorem. *Suppose D and D' are proper domains in $\mathbb{R}^2$. If $f : D \longrightarrow D'$ is K-qc and harmonic, then it is bilipschitz with respect to quasihyperbolic metrics on D and D'.*

PROOF. Since f is harmonic we have locally, representation

$$f(z) = g(z) + \overline{h(z)},$$

where g and h are analytic functions. Then Jacobian $J_f(z) = |g'(z)|^2 - |h'(z)|^2 > 0$ (note that $g'(z) \neq 0$).

Futher,

$$J_f(z) = |g'(z)|^2 \left(1 - \frac{|h'(z)|^2}{|g'(z)|^2}\right) = |g'(z)|^2 \left(1 - |\omega(z)|^2\right),$$

where $\omega(z) = \frac{h'(z)}{g'(z)}$ is analytic and $|\omega| < 1$. Now we have

$$\log \frac{1}{J_f(z)} = -2 \log |g'(z)| - \log(1 - |\omega(z)|^2).$$

The first term is harmonic function (it is well known that logarithm of moduli of analytic function is harmonic everywhere except where that analytic function vanishes, but $g'(z) \neq 0$ everywhere).

The second term can be expanded in series

$$\sum_{k=1}^{\infty} \frac{|\omega(z)|^{2k}}{k},$$

and each term is subharmonic (note that ω is analytic).

So, $-\log(1 - |\omega(z)|^2)$ is a continuous function represented as a locally uniform sum of subharmonic functions. Thus it is also subharmonic.

Hence

$$\log \frac{1}{J_f(z)} \text{ is a subharmonic function.} \tag{4.2}$$

Note that representation $f(z) = g(z) + \overline{h(z)}$ is local, but that suffices for our conclusion (4.2).

By the definition from [AG, Definition 1.5]

$$\alpha_f(z) = \exp\left(\frac{1}{n} (\log J_f)_{B_z}\right),$$

where

$$(\log J_f)_{B_z} = \frac{1}{m(B_z)} \int_{B_z} \log J_f \, dm, \quad B_z = B(z, d(z, \partial D)).$$

In the case $n = 2$ we have

$$\frac{1}{\alpha_f(z)} = \exp\left(\frac{1}{2} \frac{1}{m(B_z)} \int_{B_z} \log \frac{1}{J_f(w)} \, dm(w)\right). \tag{4.3}$$

From (4.2) we have

$$\frac{1}{m(B_z)} \int_{B_z} \log \frac{1}{J_f(w)} \, dm(w) \geq \log \frac{1}{J_f(z)}.$$

Combining this with (4.3) we have

$$\frac{1}{\alpha_f(z)} \geq \exp\left(\frac{1}{2}\log\frac{1}{J_f(z)}\right) = \frac{1}{\sqrt{J_f(z)}}$$

and therefore

$$\sqrt{J_f(z)} \geqslant \alpha_f(z).$$

On the other hand, we have Theorem [AG, Theorem 1.8]:

Suppose that D and D' are domains in $\mathbb{R}^n$ if $f : D \longrightarrow D'$ is K-qc, then

$$\frac{1}{c}\frac{d(f(x),\partial D')}{d(x,\partial D)} \leq \alpha_f(z) \leq c\,\frac{d(f(x),\partial D')}{d(x,\partial D)}$$

for $x \in D$, where c is a constant wich depends only on K and n.

From first inequality of this theorem we have

$$\sqrt{J_f(z)} \geq \frac{1}{c}\frac{d(f(x),\partial D')}{d(x,\partial D)}. \tag{4.4}$$

Note that

$$J_f(z) = |g'(z)|^2 - |h'(z)|^2 \leq |g'(z)|^2$$

and by K-qclity of f, $|h'| \leq k|g'|$, $0 \leq k < 1$, where $K = \frac{1+k}{1-k}$.

This gives $J_f \geq (1-k^2)|g'|^2$. Hence,

$$\sqrt{J_f} \asymp |g'| \asymp |g'| + |h'| = ||f'(z)||.$$

Finally (4.4) and the above asymptotic relation give

$$||f'(z)|| \geq \frac{1}{c}\frac{d(f(x),\partial D')}{d(x,\partial D)}, \quad c = c(k).$$

For the reversed inequality we again use $J_f(z) \geq (1-k^2)|g'(z)|^2$, i.e.

$$\sqrt{J_f(z)} \geq \sqrt{1-k^2}\,|g'(z)| \tag{4.5}$$

Further, we know that for $n = 2$

$$\alpha_f(z) = \exp\left(\frac{1}{m(B_z)}\int_{B_z} \log\sqrt{J_f(x)}\,dm(w)\right).$$

Using (4.5)

$$\begin{aligned}
\frac{1}{m(B_z)}\int_{B_z} \log\sqrt{J_f(x)}\,dm(w) &\geq \frac{1}{m(B_z)}\int_{B_z} \log\sqrt{1-k^2} + \log|g'(w)|\,dm(w) \\
&= \log\sqrt{1-k^2} + \frac{1}{m(B_z)}\int_{B_z} \log|g'(w)|\,dm(w) \\
&= \log\sqrt{1-k^2} + \log|g'(z)|.
\end{aligned}$$

Now we have by harmonicity of $\log|g'|$

$$\begin{aligned}
\alpha_f(z) &= \exp\left(\frac{1}{m(B_z)}\int_{B_z}\log\sqrt{J_f(x)}\,dm(w)\right)\\
&\geq \exp(\log\sqrt{1-k^2}+\log|g'(z)|)\\
&= \sqrt{1-k^2}|g'(z)|\\
&\geq \frac{1}{2}\sqrt{1-k^2}(|g'|+|h'|)\\
&= \frac{\sqrt{1-k^2}}{2}||f'||.
\end{aligned}$$

Again using the second inequality in [AG, Theorem 1.8]

$$||f'|| \leqslant c\sqrt{J_f(z)} \leqslant c\,\alpha_f(z) \leqslant c\,\frac{d(f(z),\partial D')}{d(z,\partial D)}, \quad c=c(k).$$

Summarizing

$$||f'(z)|| \asymp \frac{d(f(z),\partial D')}{d(z,\partial D)}.$$

This pointwise result, via integration along curves, easily gives

$$k_{D'}(f(z_1),f(z_2)) \asymp k_D(z_1,z_2).$$

□

4.6. Problem. Is Theorem 4.1 true in dimensions $n \geq 3$?

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